Fuzzy Hv-submodules in Γ-Hv-modules

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Abstract— In this paper, we introduced the concept of Γ-Hv-modules, which is a generalization of Γ -modules and Hv-modules. The notion of $(\in, \in \vee q)$-fuzzy Hv-submodules of a Γ-Hv-module is provided and some related properties are investigated.

Keywords— ***Hyperoperation, Algebraic hyperstructure, Γ-Hv-module,*** $(\in, \in \vee q)$***-fuzzy Hv-submodule***.

I. INTRODUCTION

The algebraic hyperstructure is a natural generalization of the usual algebraic structures which was first initiated by Marty [1]. After the pioneering work of F. Marty, algebraic hyperstructures have been developed by many researchers. A short review of which appears in [2]. A recent book on hyperstructures [3] points out their applications in geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence and probabilities. Vougiouklis [4] introduced a new class of hyperstructures so-called Hv-structure, and Davvaz [5] surveyed the theory of Hv-structures. The Hv-structures are hyperstructures where equality is replaced by non-empty intersection.

The concept of fuzzy sets was first introduced by Zadeh [6] and then fuzzy sets have been used in the reconsideration of classical mathematics. In particular, the notion of fuzzy subgroup was defined by Rosenfeld [7] and its structure was thereby investigated. Liu [8] introduced the notions of fuzzy subrings and ideals. Using the notion of "belongingness $(\in)$" and "quasi-coincidence (q)" of fuzzy points with fuzzy sets, the concept of (α, β)-fuzzy subgroup where α, β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$ was introduced in [9]. The most viable generalization of Rosenfeld's fuzzy subgroup is the notion of $(\in, \in \vee q)$ -fuzzy subgroups, the detailed study of which may be found in [10]. The concept of an $(\in, \in \vee q)$-fuzzy subring and ideal of a ring have been introduced in [11] and the concept of $(\in, \in \vee q)$-fuzzy subnear-ring and ideal of a near-ring have been introduced in [12]. Fuzzy sets and hyperstructures introduced by Zadeh and Marty, respectively, are now studied both from the theoretical point of view and for their many applications. The relations between fuzzy sets and hyperstructures have been already considered by many authors. In [13-15], Davvaz applied the concept of fuzzy sets to the theory of algebraic hyperstructures and defined fuzzy Hv-subgroups, fuzzy Hv-ideals and fuzzy Hv-submodules, which are generalizations of the concepts of Rosenfeld's fuzzy subgroups, fuzzy ideals and fuzzy submodules. The concept of a fuzzy Hv-ideal and Hv-subring has been studied further in [16, 17].The concept of (∈, ∈ ∨q)-fuzzy subhyperquasigroups of hyperquasigroups was introduced by Davvaz and Corsini [18]; also see [19-24]. As is well known, the concept of Γ –rings was first introduced by Nobusawa in 1964, which is a generalization of the concept of rings. Davvaz et.al. [25] introduced the notion of $(\in, \in \vee q)$-fuzzy Hv-ideals of a Γ-Hv-ring. This paper continues this line of research for $(\in, \in \vee q)$-fuzzy Hv-submodules of a Γ-Hv-module.

The paper is organized as follows: In Section 2, we recall some basic definitions of Hv-modules. In Sections 3 and 4, we introduce the concept of Γ-Hv-modules and present some operations of fuzzy sets in Γ-Hv-modules. In Section 5, by using a new idea, we introduce and investigate the (∈, ∈ ∨q)-fuzzy Hv-submodules of a Γ-Hv-module.

the introduction of the paper should explain the nature of the problem, previous work, purpose, and the contribution of the paper. The contents of each section may be provided to understand easily about the paper.

II. BASIC DEFINITIONS

We first give some basic definitions for proving the further results.

Definition 2.1[27] Let X be a non-empty set. A mapping $\mu : X \to [0,1]$ is called a fuzzy set in X. The complement of μ, denoted by μ^c, is the fuzzy set in X given by $\mu^c(x) = 1 - \mu(x) \quad \forall x \in X.$

Definition 2.2[28] Let G be a non–empty set and $* : G \times G \to \wp^*(G)$ be a hyperoperation, where $\wp^*(G)$ is the set of all the non-empty subsets of G. Where $A * B = \bigcup_{a\in A,\, b\in B} a * b, \ \forall A, B \subseteq G.$

The $*$ is called weak commutative if $x * y \cap y * x \neq \phi, \forall x, y \in G.$

The $*$ is called weak associative if $(x*y)*z \cap x*(y*z) \neq \phi, \forall x, y, z \in G.$

A hyperstructure $(G,*)$ is called an Hv-group if
(i) $*$ is weak associative.
(ii) $a*G = G*a = G, \quad \forall a \in G$ (Reproduction axiom).

Definition 2.3[28] An Hv-ring is a system $(R,+,\cdot)$ with two hyperoperations satisfying the ring-like axioms:

(i) $(R,+)$ is an Hv-group, that is,

$((x+y)+z) \cap (x+(y+z)) \neq \phi \quad \forall x, y \in R,$

$a+R = R+a = R \quad \forall a \in R;$

(ii) $(R,\cdot)$ is an Hv-semigroup;

(iii) $(\cdot)$ is weak distributive with respect to $(+)$, that is, for all $x, y, z \in R$

$(x\cdot(y+z)) \cap (x\cdot y + x\cdot z) \neq \phi,$

$((x+y)\cdot z) \cap (x\cdot z + y\cdot z) \neq \phi.$

Definition 2.4[26] Let R be an Hv-ring. A nonempty subset I of R is called a left (resp., right) Hv-ideal if the following axioms hold:
(i) $(I,+)$ is an Hv-subgroup of $(R,+)$,
(ii) $R\cdot I \subseteq I$ (resp., $I\cdot R \subseteq I$).

Definition 2.5[26] Let $(R,+,\cdot)$ be an Hv–ring and μ a fuzzy subset of R. Then μ is said to be a left (resp., right) fuzzy H_v-ideal of R if the following axioms hold: $(1) \min\{\mu(x), \mu(y)\} \leq \inf\{\mu(z) : z \in x+y\} \forall x, y \in R,$
(2) For all $x, a \in R$ there exists $y \in R$ such that $x \in a+y$ and $\min\{\mu(a), \mu(x)\} \leq \mu(y),$
(3) For all $x, a \in R$ there exists $z \in R$ such that $x \in z+a$ and $\min\{\mu(a), \mu(x)\} \leq \mu(z),$ $(4) \mu(y) \leq \inf\{\mu(z) : z \in x\cdot y\}$ respectively $\mu(x) \leq \inf\{\mu(z) : z \in x\cdot y\} \quad \forall x, y \in R.$

Let μ be a fuzzy subset of a non-empty set X and let t ∈ (0, 1]. The set $\mu_t = \{x \in X \mid \mu(x) \geq t\}$ is called a level cut of μ.
Theorem 2.2[15] Let R be an Hv-ring and μ a fuzzy subset of R. Then μ is a fuzzy left (resp., right) Hv-ideal of R if and only if for every t ∈ (0, 1], $\mu_t (\neq \phi)$ is a left (resp., right) Hv-ideal of R.

When μ is a fuzzy Hv-ideal of R, μ_t is called a level Hv-ideal of R. The concept of level Hv-ideal has been used extensively to characterize various properties of fuzzy Hv-ideals.

III. Γ-HV-MODULES

The concept of Γ -rings was introduced by Nobusawa in 1964.
Definition 3.1[29] Let (R, +) and (Γ ,+) be two additive abelian groups. Then R is called a Γ -ring if the following conditions are satisfied for all x, y, z ∈ R and for all α, β ∈ Γ,
(1) xαy ∈ R,
(2) (x + y)αz = xαz + yαz, x(α + β)y = xαy + xβy, xα(y + z) = xαy + xαz,
(3) xα(yβz) = (xαy)βz.
Definition 3.2[25] Let (R, ⊕) and (Γ , ⊕) be two Hv-groups. Then R is called a Γ –Hv-ring if the following conditions are satisfied for all x, y, z ∈ R and for all α, β ∈ Γ,

(1) $x\alpha y \subseteq R,$

$(2) (x \oplus y)\alpha z \cap (x\alpha z \oplus y\alpha z) \neq \phi, x(\alpha \oplus \beta)y \cap (x\alpha y \oplus x\beta y) \neq \phi, x\alpha(y \oplus z) \cap (x\alpha y \oplus x\alpha z) \neq \phi,$

(3) $x\alpha(y\beta z) \cap (x\alpha y)\beta z \neq \phi.$

In what follows, unless otherwise stated, (R, ⊕, Γ) always denotes a Γ –Hv-ring.
Definition 3.3[25] A subset I in R is said to be a left (resp., right) Hv-ideal of R if it satisfies
(1) (I, ⊕) is an Hv-subgroup of (R, ⊕),
(2) xαy ⊆ I (resp., yαx ⊆ I) for all x ∈ R, y ∈ I and α ∈ Γ .

I is said to be an Hv-ideal of R if it is both a left and a right Hv-ideal of R.
Now, we introduce the concept of Γ –Hv-modules as follows.

Definition 3.4 Let $(R,\oplus,\Gamma)$ be a Γ-H_v-ring and $(M,\oplus)$ be a canonical H_v-group. M is called a Γ-H_v-module over R if there exists $f: R\times\Gamma\times M \to M$ (the image of (r,α,m) being denoted by $r\alpha m$) such that for all $a,b\in R, m_1,m_2\in M, \alpha,\beta\in\Gamma,$ we have

$(1)\, a\alpha(m_1\oplus m_2)\cap a\alpha m_1\oplus a\alpha m_2\neq\phi;$

$(2)\, (a\oplus b)\alpha m_1\cap a\alpha m_1\oplus b\alpha m_1\neq\phi;$

$(3)\, a(\alpha\oplus\beta)m_1\cap a\alpha m_1\oplus a\beta m_1\neq\phi;$

$(4)\, (a\alpha b)\beta m_1\cap a\alpha(b\beta m_1)\neq\phi.$

Throughout this paper, R and M are a Γ-H_v-ring and Γ-H_v-module, respectively, unless otherwise specified.

Definition 3.5 A subset A in M is said to be a Γ-H_v-submodule of M if it satisfies the following conditions:

$(1)\, (A,\oplus)$ is a H_v-subgroup of $(M,\oplus)$;

$(2)\, r\alpha x\in A,$ for all $r\in R, \alpha\in\Gamma$ and $x\in A.$

4. Fuzzy sets in Γ-Hv-modules

Let X be a non-empty set. The set of all fuzzy subsets of X is denoted by $F(X).$ For any $A\subseteq X$ and r ∈ (0, 1], the fuzzy subset r_A of X is defined by

$$r_A(x)=\begin{cases} r & if\ x\in A,\\ 0 & otherwise,\end{cases}$$

for all $x\in X.$ In particular, when r = 1, r_A is said to be the characteristic function of A, denoted by $\chi_A;$ when A = {x}, r_A is said to be a fuzzy point with support x and value r and is denoted by $x_r.$ A fuzzy point x_r is said to belong to (resp., be quasicoincident with) a fuzzy set $\mu,$ written as $x_r\in\mu$ (resp., $x_r q\mu$) if $\mu(x)\geq r$ (resp., $\mu(x)+r>1$). If $\mu(x)\geq r$ or $\mu(x)+r>1$, then we write $x_r\in\vee q\mu.$

For $\mu\in F(X),$ μ is said to have the sup-property if for any non-empty subset A of X, there exists $x\in A$ such that $\mu(x)=\vee_{y\in A}\mu(y).$

Next we define a new ordering relation “$\subseteq\vee q$” on $F(X),$ which is called the fuzzy inclusion or quasi-coincidence relation, as follows:

For any $\mu,\nu\in F(X), \mu\subseteq\vee q\nu$ if and only if $x_r\in\mu$ implies $x_r\in\vee q\mu$ for all $x\in X$ and $r\in(0,1].$ And we define the relation “≈” on F(X) as follows:

For any $\mu,\nu\in F(X), \mu\approx\nu$ if and only if $\mu\subseteq\vee q\nu$ and $\nu\subseteq\vee q\mu.$

In what follows, unless otherwise stated, $M(r_1,r_2,.....r_n),$ where n is a positive integer, will denote $r_1\wedge r_2\wedge.....\wedge r_n$ for all $r_1,r_2,.....r_n\in[0,1],$ $\overline{\in\vee q}$ means $\in\vee q$ does not hold and $\overline{\subseteq\vee q}$ implies $\subseteq\vee q$ is not true.

Lemma 4.1[30] Let X be a non-empty set and μ, ν ∈ F(X). Then μ ⊆ ∨ qν if and only if ν(x) ≥ M (μ(x), 0.5) for all x ∈ X.
Lemma 4.2[30] Let X be a non-empty set and μ, ν, ω ∈ F(X) be such that μ ⊆ ∨ qν ⊆ ∨ qω. Then μ ⊆ ∨ qω.
Clearly, Lemma 4.1 implies that μ ≈ ν if and only if M(μ(x), 0.5) = M(ν(x), 0.5) for all x ∈ X and μ, ν ∈ F(X), and it follows from Lemmas 4.1 and 4.2 that “≈” is an equivalence relation on F(X).
Now, let us define some operations of fuzzy subsets in a Γ –Hv-module M.

Definition 4.3. Let μ, ν ∈ F(M) and α ∈ Γ. We define fuzzy subsets $\mu\oplus\nu$ and $\mu\overline{\alpha}\nu$ by

$(\mu\oplus\nu)(z)=\underset{z\in x\otimes y}{\vee} M(\mu(x),\nu(y))$ and $(\mu\overline{\alpha}\nu)(z)=\underset{z\in x\alpha y}{\vee} M(\mu(x),\nu(y))$ respectively, for all z ∈ M and α ∈ Γ.

It is worth noting that for any $x_r, y_s\in F(M), x_r\oplus y_s=M(r,s)_{x\otimes y}$ and $x_r\overline{\alpha}y_s=M(r,s)_{x\alpha y}$.

Lemma 4.4. Let $\mu_1,\mu_2,\nu_1,\nu_2\in F(R)$ such that $\mu_1\subseteq\vee q\mu_2$ and $\nu_1\subseteq\vee q\nu_2$. Then $(1)\,\mu_1\oplus\nu_1\subseteq\vee q\mu_2\oplus\nu_2$ and $\mu_1\overline{\alpha}\nu_1\subseteq\vee q\mu_2\overline{\alpha}\nu_2$ for all α ∈ Γ.

$(2)\,\mu_1\cap\nu_1\subseteq\vee q\mu_2\cap\nu_2$.

Lemma 4.4 indicates that the equivalence relation ‘‘≈’’ is a congruence relation on $(F(R),\oplus)$ and $(F(R),\overline{\alpha})$ for all α ∈ Γ.

Lemma 4.5. Let μ, ν, ω ∈ F(R). Then

$(1)\,\mu\oplus(\nu\cup\omega)=\mu\oplus\nu\cup\mu\oplus\omega, (\mu\cup\nu)\oplus\omega=\mu\oplus\omega\cup\nu\oplus\omega.$

$(2)\,\mu\oplus(\nu\cap\omega)\subseteq\mu\oplus\nu\cap\mu\oplus\omega, (\mu\cap\nu)\oplus\omega\subseteq\mu\oplus\omega\cap\nu\oplus\omega.$

$(3)\,\mu\overline{\alpha}(\nu\cup\omega)=\mu\overline{\alpha}\nu\cup\mu\overline{\alpha}\omega, (\mu\cup\nu)\overline{\alpha}\omega=\mu\overline{\alpha}\omega\cup\nu\overline{\alpha}\omega$ for all α ∈ Γ.

$(4)\,\mu\overline{\alpha}(\nu\cap\omega)\subseteq\mu\overline{\alpha}\nu\cap\mu\overline{\alpha}\omega, (\mu\cap\nu)\overline{\alpha}\omega=\mu\overline{\alpha}\omega\cap\nu\overline{\alpha}\omega$ for all α ∈ Γ. $(5)\,\mu\overline{\alpha}(\nu\oplus\omega)\subseteq\mu\overline{\alpha}\nu\oplus\mu\overline{\alpha}\omega$ for all α ∈ Γ.

Proof. The proof of (1)–(4) is straightforward. We show (5). Let x ∈ R and α ∈ Γ. If $x\notin y\alpha z$ for all y, z ∈ R. Then $(\mu\overline{\alpha}(\nu\oplus\omega))(x)=0\le(\mu\overline{\alpha}(\nu\oplus\omega))(x).$ Otherwise, we have

$$(\mu\overline{\alpha}(\nu\oplus\omega))(x)=\underset{x\in y\alpha z}{\vee} M(\mu(y),(\nu\oplus\omega))(z)=\underset{x\in y\alpha z}{\vee} M\Big(\mu(y),\underset{z\in p\otimes q}{\vee} M(\nu(p),\omega(q))\Big)$$

$$=\underset{x\in y\alpha z, z\in p\otimes q}{\vee} M(\mu(y),\nu(p),\mu(y),\omega(q))\le\underset{x\in a\otimes b, a\in y\alpha p, b\in y\alpha q}{\vee} M\big((\mu\overline{\alpha}\nu)(a),(\mu\overline{\alpha}\omega)(b)\big)$$

$$\le\underset{x\in y'\otimes z'}{\vee} M\big((\mu\overline{\alpha}\nu)(y'),(\mu\overline{\alpha}\omega)(z')\big)=(\mu\overline{\alpha}\nu\oplus\mu\overline{\alpha}\omega)(x).$$

Hence $\mu\overline{\alpha}(\nu\oplus\omega)\subseteq\mu\overline{\alpha}\nu\oplus\mu\overline{\alpha}\omega.$ This completes the proof.

IV. (∈, ∈ ∨Q)-FUZZY HV-SUBMODULES OF A Γ –HV-MODULE

In this section, using the new ordering relation on F (M), we define and investigate (∈, ∈ ∨q)-fuzzy left (right) Hv-submodules of Γ –Hv-module.

Definition 5.1. A fuzzy subset μ of M is called an (∈, ∈ ∨q)-fuzzy left (resp., right) Hv-submodule if it satisfies the following conditions:

(F1a) $\mu\oplus\mu\subseteq\vee q\mu,$

(F2a) $x_t, a_s\in\mu$ implies that there exists y ∈ M such that $x\in a\otimes y$ and $y_{M(r,s)}\in\vee q\mu,$

(F3a) $x_t, a_s\in\mu$ implies that there exists z ∈ M such that $x\in z\otimes a$ and $z_{M(r,s)}\in\vee q\mu,$

(F4a) $\chi_R\overline{\alpha}\mu\subseteq\vee q\mu$ (resp., $\mu\overline{\alpha}\chi_R\subseteq\vee q\mu$) for all α ∈ Γ.

A fuzzy subset μ of M is called an (∈, ∈ ∨q)-fuzzy Hv-submodule of M if it is both an (∈, ∈ ∨q)-fuzzy left and an (∈, ∈ ∨q)-fuzzy right Hv-submodule of M.

Before proceeding, let us first provide some auxiliary lemmas.

Lemma 5.2. Let μ ∈ F (M). Then (F1a) holds if and only if one of the following conditions holds:

(F1b) $x_r, y_s\in\mu$ implies $M(r,s)_{x\oplus y}\subseteq\vee q\mu$ for all x, y ∈ M and r, s ∈ (0, 1].

(F1c) $\underset{z\in x\oplus y}{\wedge}\mu(z)\ge M(\mu(x),\mu(y),0.5)$ for all x, y ∈ M.

Proof. (F1a)⇒ (F1b) Let x, y ∈ M and r, s ∈ (0, 1] be such that $x_r, y_s \in \mu$. Then for any $z \in x \oplus y,$ we have

$$(\mu \oplus \mu)(z) = \vee_{z \in a \oplus b} M(\mu(a), \mu(b)) \geq M(\mu(x), \mu(y)) \geq M(r, s).$$

Hence $z_{M(r,s)} \in \mu \oplus \mu$ and so $M(r,s)_{x \oplus y} \subseteq \vee q\mu.$ It follows from (F1a) that $M(r,s)_{x \oplus y} \subseteq \vee q\mu.$

(F1b) ⇒ (F1c) Let x, y ∈ M. If possible, let z ∈ M be such that $z \in x \oplus y$ and μ (z) < r = M (μ(x), μ(y), 0.5). Then $x_r, y_r \in \mu$ and μ (z) + r < r + r ≤ 1, that is, $z_r \overline{\in \vee q} \mu,$ a contradiction. Hence (F1c) is valid.

(F1c)⇒ (F1a) For $x_r \in \mu \oplus \mu,$ if possible, let $x_r \overline{\in \vee q} \mu,$ Then μ(x) < r and μ(x) < 0.5. If $x \in y \oplus z$ for some y, z ∈ M, by (F1c), we have 0.5 > μ(x) ≥ M (μ(y), μ(z), 0.5), which implies μ(x) ≥ M(μ(y), μ(z)).

Hence we have $r \leq (\mu \oplus \mu)(x) = \vee_{x \in a \otimes b} M(\mu(a), \mu(b)) \leq \vee_{x \in a \otimes b} \mu(x) = \mu(x),$ a contradiction. Hence (F1a) is satisfied.

Lemma 5.3. Let μ ∈ F (M). Then (F2a) holds if and only if the following condition holds:

(F2b) for all x, a ∈ M there exists y ∈ M such that $x \in a \oplus y$ and M (μ (a), μ(x), 0.5) ≤ μ(y).

Proof. (F2a) ⇒ (F2b): Suppose that x, a ∈ M. We consider the following cases:

(a) M (μ(x), μ (a)) < 0.5,

(b) M (μ(x), μ (a)) ≥ 0.5.

Case a: Assume that for all y with $x \in a \oplus y$, we have μ (y) < μ (x) ∧ μ (a). Choose t such that μ(y) < t < M (μ(x), μ (a)). Then $x_t, a_t \in \mu$ but $y_t \overline{\in \vee q} \mu,$ which contradicts (F2a).

Case b: Assume that for all y with $x \in a \oplus y$, we have μ(y) < M (μ(x), μ (a), 0.5). Then $x_{0.5}, a_{0.5} \in \mu,$ but $y_{0.5} \overline{\in \vee q} \mu,$ which contradicts (F2a).

(F2b) ⇒ (F2a): Let $x_t, a_r \in \mu.$ Then μ(x) ≥ t and μ (a) ≥ r. Now, for some y with $x \in a \oplus y$, we have μ (y) ≥ M(μ(a), μ(x), 0.5) ≥ M(t, r, 0.5).

If M (t, r) > 0.5, then μ(y) ≥ 0.5 which implies μ(y) + M (t, r) > 1.

If M (t, r) ≤ 0.5, then μ(y) ≥ M (t, r).

Therefore, $y_{M(t,r)} \in \vee q\mu.$ Hence (F2a) holds.

Lemma 5.4. Let μ ∈ F(M). Then (F3a) holds if and only if the following condition holds:

(F3b) for all x, a ∈ M there exists z ∈ M such that $x \in z \oplus a$ and M (μ(a), μ(x), 0.5) ≤ μ(z).

Proof. It is similar to the proof of Lemma 5.3.

Lemma 5.5. Let μ ∈ F(M) and α ∈ Γ. Then (F4a) holds if and only if one of the following conditions holds:

(F4b) $x_r \in \mu$ implies $r_y \alpha x_1 \subseteq \vee q\mu$ (resp., $r_x \alpha y_1 \subseteq \vee q\mu$) for all x, y, z ∈ M and r ∈ (0, 1].

(F4c) $\wedge_{z \in x\alpha y} \mu(z) \geq M(\mu(y), 0.5)$ (resp., $\wedge_{z \in x\alpha y} \mu(z) \geq M(\mu(x), 0.5))$ for all x, y ∈ M and α ∈ Γ.

Proof. The proof is similar to the proof of Lemma 5.2.

Let μ be a fuzzy subset and r ∈ (0, 1]. Then the set $[\mu]_r = \{x \in R \mid x_r \in \vee q\mu\}$ is called the ∈ ∨ q-level subset of μ.

The next theorem provides the relationship between (∈, ∈ ∨ q)-fuzzy left (resp., right) Hv-submodules of Γ –Hv-module and crisp left (resp., right) Hv-submodules of Γ –Hv-module.

Theorem 5.6. Let $\mu \in F(M).$ Then

(1) μ is an $(\in, \in \vee q)$-fuzzy left (resp., right) Hv-submodule of M if and only if $\mu_r (\mu_r \neq \phi)$ is a left (resp., right) Hv-submodule of M for all $r \in (0, 0.5].$

(2) μ is an $(\in, \in \vee q)$-fuzzy left (resp., right) Hv-ideal of R if and only if $[\mu]_r ([\mu]_r \neq \phi)$ is a left (resp., right) Hv-ideal of R for all $r \in (0,1].$

Proof. We only show (2). Let μ be an $(\in, \in \vee q)$-fuzzy left Hv-submodule of M and $x, y \in [\mu]_r$ for some $r \in (0,1]$. Then μ(x) ≥ r or μ(x) > 1 − r and μ(y) ≥ r or μ(y) > 1 − r. Since μ is an $(\in, \in \vee q)$-fuzzy left Hv-submodule of M, we have μ(z) ≥ M(μ(x), μ(y), 0.5) for all $z \in x \oplus y$. We consider the following cases:

(a) $r \in (0, 0.5]$.

(b) $r \in (0.5, 1]$.

Case a: Since $r \in (0, 0.5]$. We have 1 − r ≥ 0.5 ≥ r and so μ(z) ≥ M(r, r, 0.5) = r or μ(z) ≥ M(r, 1 − r, 0.5) = r or μ(z) ≥ M(1 − r, 1 − r, 0.5) = 0.5 ≥ r for all $z \in x \oplus y$. Hence $z_r \in \mu$ for all $z \in x \oplus y$.

Case b: Since $r \in (0.5, 1]$. We have 1−r < 0.5 < r and so μ(z) ≥ M(r, r, 0.5) = 0.5 or μ(z) > M(r, 1−r, 0.5) = 1−r or μ(z) > M(1 − r, 1 − r, 0.5) = 1 − r for all $z \in x \oplus y$. Hence $z_r \in \vee q \mu$ for all $z \in x \oplus y$. Thus in any case, $z \in [\mu]_r$ for all $z \in x \oplus y$. Similarly we can show that there exist $y, z \in [\mu]_r$ such that $x \in a \oplus y$ and $x \in z \oplus a$ for all $x, a \in [\mu]_r$ and that $x \alpha y \in [\mu]_r$ for all $x \in M$, $y \in [\mu]_r$ and $\alpha \in \Gamma$. Therefore, $[\mu]_r$ is a left Hv-submodule of M.

Conversely, let $\mu \in F(M)$ and $[\mu]_r$ is a left Hv-submodule of M for all $r \in (0,1]$. Let $x, y \in M$. If there exists $z \in M$ such that μ(z) < r = M(μ(x), μ(y), 0.5), then $x, y \in [\mu]_r$ but $z \notin [\mu]_r$, a contradiction. Therefore, μ(z) ≥ M(μ(x), μ(y), 0.5) for all $z \in x \oplus y$. Similarly we can show that there exist $y, z \in M$ such that $x \in a \oplus y, x \in z \oplus a$, μ(y) ≥ M(μ(x), μ(a), 0.5) and μ(z) ≥ M(μ(x), μ(a), 0.5) and that μ(xαy) ≥ M(μ(y), 0.5) for all $x, y \in M$ and $\alpha \in \Gamma$. Therefore, μ is an $(\in, \in \vee q)$-fuzzy left Hv-submodule of M by Lemmas 5.2–5.5.

The case for $(\in, \in \vee q)$-fuzzy right Hv-submodules of M can be similarly proved.

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