# The function of Green for the bioheat equation of Pennes in an axisymmetric unbounded domain

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***Abstract*— *The function of Green associated to a linear partial differential operator P(D) in a domain* $\Omega$ *acting at point* $\mathbf{x}_0$ *of the domain, is a distribution* $G(\mathbf{x},\mathbf{x}_0)$ *such that* $P(D)G(\mathbf{x},\mathbf{x}_0) = \delta(\mathbf{x}-\mathbf{x}_0)$*, where δis the Dirac's delta distribution. The property* $P(D)G(\mathbf{x},\mathbf{x}_0) = \delta(\mathbf{x}-\mathbf{x}_0)$ *of a Green's function can be exploited to solve differential equations of the form P(D)u = f, because*

$$\int_\Omega P(D)G(\mathbf{x},\mathbf{x}_0)f(\mathbf{x}_0)d\mathbf{x}_0 = f(\mathbf{x}) = P(D)u,$$

*Hence*

$$P(D)u = P(D)(\int_\Omega G(\boldsymbol{x},\boldsymbol{x_0})f(\boldsymbol{x_0})d\boldsymbol{x_0})$$

*which implies that* $u = G(\mathbf{x},\mathbf{x}_0)f(\mathbf{x}_0)d\mathbf{x}_0$*. Not every operator P(D) admits a Green's function. And the Green's function, if it exists, is not unique, but adding boundary conditions it will be unique. In regular Sturm-Liouville problems, there is an standard way to obtain the corresponding Green's function, and after that, as the domain is bounded, to incorporate the initial and boundary conditions using also the Green's function. But the method doesn't work if the domain is not bounded, because the justification is based in the use of the Green's Theorem. In this paper we find the Green's function for the Pennes's bioheat equation, see [1], in a unbounded domain consisting in the space* $R^3$ *with an infinite cylindrical hole. This type of problems appears in radiofrequency (RF) ablation with needle-like electrodes, which is widely used for medical techniques such as tumor ablation or cardiac ablation to cure arrhythmias. We recall that theoretical modeling is a rapid and inexpensive way of studying different aspects of the RF process.*

## I. INTRODUCTION

In the theory of heat conduction in perfused biological tissues, the so called Pennes's bioheat equation, that is

$$\eta\, c\, \frac{\partial T}{\partial t} = \nabla.\,(k\,\nabla T) + S - \eta_b c_{\ b}\omega_b(T - Tb) \tag{1}$$

plays a central role. In (1) $T(\mathbf{x},t)$ denotes the temperature at every point $\mathbf{x}$ of a biological tissue lying in a domain $\Omega \subset \mathbf{R}^3$ in the instant $t$, the (assumed constant) terms $\eta, c$ and $k$ are the density, specific heat and thermal conductivity of the tissue respectively, $\eta_b, c_b, \omega_b$ and $T_b$ are the density, specific heat, perfusion coefficient of the blood and blood temperature respectively (all assumed constant too) and $S = S(\mathbf{x},t)$ represents the heat sources.

We consider the following infinite spatial domain:

$$\Omega := \{(x,y,z)/\, x^2 + y^2 \geq r_0^2,\ \ r_0 > 0\}$$

with source $S = S(\sqrt{x^2+y^2}, t)$ bounded in $\Omega$, and initial and boundary conditions only dependent on $\sqrt{x^2+y^2}$ and the temporal variable $t$. This is the geometry used for problems related to radio frecuency ablation of tumors with needle-like electrodes. In this case Haemmerich in [2] proposed a heat source independent of the time $S(r) = \frac{j_0^2 r_0^2}{\sigma r^2}$, where $j_0$ is the current density at the conductor surface and $\sigma$ the electrical conductivity.

Switching to cylindrical coordinates the Pennes's bioheat equation (1) becomes:

$$\eta\, c\, \frac{\partial T}{\partial t} - k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) + \eta_b c_{\ b}\omega_b(T - Tb) = S(r;\ t) \tag{2}$$

The initial and boundary condition at the infinity we will consider in this paper are the following:

$$T(r,0) = g(r),\ \forall r > r_0$$

$$\lim_{r\to\infty} T(r,t) = h(t),\quad \forall t > 0$$

$$T(r_0,t) = f(t),\ \forall t > 0$$

Where $f(t)$ can be interpreted as a refrigeration temperature in the boundary of the hole. For example, RF ablation with internally needle-like electrodes is widely used for medical techniques (see the references given in [1] for example). The device consists of an internally liquid cooled metallic cylindrical electrode that cools the electrode surface. In this case $g(r) = h(\xi) = T_b$ and $f(t) = T_C$, where $T_C$ is the temperature fixed by the refrigeration of the electrode, see [3] and [4].

$$\rho = \frac{r}{r_0},\ \xi = \frac{kt}{c\eta r_0^2}\beta := \frac{\eta_b c_b w_b r_0^2}{k}$$

$$V(\rho,\xi) := \sigma\, k(T(\rho\, r_0, \xi\eta\, c\, r_0^2/k) - Tb),\ S_1(\rho,\xi) := S(\rho r_0, \xi\eta\, c\, r_0^2/k),$$

$$g_1(\rho) = \sigma\, k\,(g(\rho\, r0) - Tb),\ \ f_1(\xi) := \sigma\, k(f(\xi\eta\, c\, r_0^2/k) - Tb),$$

$$h_1(\xi) := \sigma\, k\,(h(\xi\eta\, c\, r_0^2/k)\text{-Tb}).$$

Then (2) becomes

$$\frac{\partial}{\partial\rho}\left((-\rho)\frac{\partial V}{\partial\rho}\right) + \frac{\partial V}{\partial\xi} + \rho\beta V = \rho S_1(\rho,\xi) \tag{3}$$

$$V(\rho,0) = g_1(\rho),\ \forall\rho>1$$

$$lim_{\rho\to\infty} V(\rho,\xi) = h_1(\xi),\ \forall\xi>0$$

$$V(1,\xi) = f_1(\xi),\ \forall\xi>0.$$

## II. THE GREEN'S FUNCTION

Our propose is to obtain a function $G(\rho,\rho_0,\xi,\xi_0)$ such that

$$\frac{\partial}{\partial\rho}\left((-\rho)\frac{\partial G}{\partial\rho}\right) + \frac{\partial G}{\partial\xi} + \rho\beta G = \delta(\rho-\rho_0)\ \delta(\xi-\xi_0) \tag{4}$$

$$G(1,\rho_0,\xi,\xi_0) = 0,\quad \forall\xi,\ \xi_0>0,\ \forall\rho_0>1$$

$$lim_{\rho\to\infty} G(\rho,\rho_0,\xi,\xi_0) = 0,\quad \forall\xi,\ \xi_0>0,\ \forall\rho_0>1$$

$$lim_{\xi\to\infty} G(\rho,\rho_0,\xi,\xi_0) = 0,\ \forall\xi_0>0,\ \forall\rho,\ \rho_0>1$$

and after that to prove that $G(\rho,\rho_0,\xi,\xi_0)$ is the Green's function of (11).

The Laplace transform of $G(\rho,\rho_0,\xi,\xi_0)$, denoted $L[G](\rho,\rho_0,\xi,\xi_0)$, with respect to $\xi$ verifies

$$\frac{\partial}{\partial\rho}\left((-\rho)\frac{\partial\mathfrak{L}[G]}{\partial\rho}\right) + \rho(s+\beta)\mathfrak{L}[G] = \delta(\rho-\rho_0)\ e^{-s\,\xi_0} \tag{5}$$

$$L[G](1,\rho_0,s,\xi_0) = 0$$

$$lim_{\rho\to\infty} L[G](\rho,\rho_0,s,\xi_0) = 0$$

$$\lim_{s\to 0} s\ \mathfrak{L}[G](\rho,\rho_0,s,\xi_0) = 0.$$

To solve this equation we proceed as in the regular Sturm Liouville problems, see for example [5], [8] and [6], and we begin finding the function $W(\rho,\rho_0,s,\xi_0)$ for the boundary value problem

$$\frac{\partial}{\partial \rho}\left(\left(-\rho\right)\frac{\partial W}{\partial \rho}\right)+\frac{\partial W}{\partial \xi}+\rho\beta W=0 \tag{6}$$

$$W(1,\rho_0,s,\xi_0)=0 \tag{7}$$

$$lim_{\rho\to\infty} W(\rho,\rho_0,s,\xi_0)=0 \tag{8}$$

The equation (6) is a modified Bessel equation of order 0.

Then if $\rho\leq\rho_0,$

$$W(\rho,\rho_0,s,\xi_0)=C_1(s)I_0(\rho\sqrt{s+\beta})+C_2(s)K_0(\rho\sqrt{s+\beta})$$

$$W(\rho,\rho_0,s,\xi_0)=C_1(s)I_0(\rho\sqrt{s+\beta})+C_2(s)K_0(\rho\sqrt{s+\beta})$$

And if $\rho_0\leq\rho$

$$W(\rho,\rho_0,s,\xi_0)=C_3(s)I_0(\rho\sqrt{s+\beta})+C_4(s)K_0(\rho\sqrt{s+\beta}) \tag{9}$$

By (8) we have $C_3(s) = 0$, by (7)

$$C_1(s)I_0(\sqrt{s+\beta})+C_2(s)K_0(\sqrt{s+\beta})=0$$

And by the remaining conditions

$$C_1(s)I_0(\rho_0\sqrt{s+\beta})+C_2(s)K_0(\rho_0\sqrt{s+\beta})-C_4(s)K_0(\rho_0\sqrt{s+\beta})=0$$

$$C_4(s)K_0'(\rho_0\sqrt{s+\beta})-C_1(s)I_0'(\rho_0\sqrt{s+\beta})-C_2(s)K_0'(\rho_0\sqrt{s+\beta})=-\frac{e^{-s\xi_0}}{\rho_0\sqrt{s+\beta}}$$

Solving that system and having in mind that

$I_0(x)K_0'(x)-I_0'(x)K_0(x)=-\frac{1}{x}$

It is obtained

$$C_1(s)=e^{-s\xi_0}K_0(\rho_0\sqrt{s+\beta})$$

$$C_2(s)=-\frac{e^{-s\xi_0}K_0(\rho_0\sqrt{s+\beta})I_0(\sqrt{s+\beta})}{K_0(\rho_0\sqrt{s+\beta})}$$

$$C_4(s)=-\frac{e^{-s\xi_0}}{K_0(\sqrt{s+\beta})}\left(I_0(\sqrt{s+\beta})K_0(\rho_0\sqrt{s+\beta})+K_0(\sqrt{s+\beta})I_0(\rho_0\sqrt{s+\beta})\right)$$

$$W(\rho,\rho_0,s,\xi_0)=e^{-s\xi_0}\times$$

$$\begin{cases} I_0(\rho\sqrt{s+\beta})K_0(\rho_0\sqrt{s+\beta})-\frac{K_0(\rho\sqrt{s+\beta})K_0(\rho_0\sqrt{s+\beta})I_0(\sqrt{s+\beta})}{K_0(\sqrt{s+\beta})} & \text{If } \rho_0\geq\rho>1 \\ I_0(\rho_0\sqrt{s+\beta})K_0(\rho\sqrt{s+\beta})-\frac{K_0(\rho\sqrt{s+\beta})K_0(\rho_0\sqrt{s+\beta})I_0(\sqrt{s+\beta})}{K_0(\sqrt{s+\beta})} & \text{If } 1<\rho_0\leq\rho \end{cases} \tag{10}$$

Our score is to prove that although the problem is not an Sturm Liouville one, the corresponding Green's function is $L^{-1}[W](\rho,\rho_0,\xi,\xi_0)$. That is, we have to show that the function

$$V_1(\rho,\xi) = \int_1^{\infty}\left(\int_0^{\xi}\mathfrak{L}^{-1}[W](\rho,\rho_0,\xi,\xi_0)(\rho_0 S_1(\rho_0,\xi_0))d\xi_0\right)d\rho_0$$

is the solution of the problem:

$$\frac{\partial}{\partial\rho}\left((-\rho)\frac{\partial V_H}{\partial\rho}\right) + \frac{\partial V_H}{\partial\xi} + \rho\beta V_H = \rho S_1(\rho,\xi) \tag{11}$$

$$V_H(\rho,0) = 0, \quad \forall\rho > 1$$

$$\lim_{\rho\to\infty} V_H(\rho,\xi) = 0, \quad \forall\xi > 0$$

$$V_H(1,\xi) = 0,\ \forall\xi > 0.$$

Recall that if $\rho_0 \geq \rho > 1$,

$$L^{-1}[W](\rho,\rho_0,\xi,\xi_0) = L^{-1}[I_0(\rho\sqrt{s+\beta})\, K_0(\rho_0\sqrt{s+\beta}) - K_0(\rho\sqrt{s+\beta})K_0(\rho_0\sqrt{s+\beta})I_0(\sqrt{s+\beta}) / K_0(\rho\sqrt{s+\beta})]\ (\rho,\rho_0,\xi-\xi_0)$$

And if $\rho \geq \rho_0 > 1$,

$$L^{-1}[W](\rho,\rho_0,\xi,\xi_0) = L^{-1}[I_0(\rho_0\sqrt{s+\beta})\, K_0(\rho\sqrt{s+\beta}) - K_0(\rho\sqrt{s+\beta})K_0(\rho_0\sqrt{s+\beta})I_0(\sqrt{s+\beta}) / K_0(\rho\sqrt{s+\beta})]\ (\rho,\rho_0,\xi-\xi_0).$$

We denote:

$$W_1(\rho,\rho_0,s) = I_0(\rho_0\sqrt{s+\beta})K_0(\rho\sqrt{s+\beta}) - \frac{K_0(\rho\sqrt{s+\beta})K_0(\rho_0\sqrt{s+\beta})I_0(\sqrt{s+\beta})}{K_0(\sqrt{s+\beta})}$$

$$W_2(\rho,\rho_0,s) = I_0(\rho\sqrt{s+\beta})K_0(\rho_0\sqrt{s+\beta}) - \frac{K_0(\rho\sqrt{s+\beta})K_0(\rho_0\sqrt{s+\beta})I_0(\sqrt{s+\beta})}{K_0(\sqrt{s+\beta})}$$

Then,

$$\int_1^{\infty}\left(\int_0^{\xi}\mathfrak{L}^{-1}[W](\rho,\rho_0,\xi,\xi_0)(\rho_0 S_1(\rho_0,\xi_0))d\xi_0\right)d\rho_0 =$$

$$\int_1^{\infty}\left(\int_0^{\xi}\mathfrak{L}^{-1}[W_2](\rho,\rho_0,\xi,\xi_0)S_1(\rho_0,\xi_0)d\xi_0\right)\rho_0 d\rho_0 =$$

$$\int_\rho^{\infty}\left(\int_0^{\xi}\mathfrak{L}^{-1}[W_1](\rho,\rho_0,\xi,\xi_0)S_1(\rho_0,\xi_0)d\xi_0\right)\rho_0 d\rho_0 +$$

$$\int_1^{\rho}\left(\int_0^{\xi}\mathfrak{L}^{-1}[W_2](\rho,\rho_0,\xi,\xi_0)S_1(\rho_0,\xi_0)d\xi_0\right)\rho_0 d\rho_0 =$$

$$\int_\rho^{\infty}\left(\int_0^{\xi}\mathfrak{L}^{-1}[I_0(\rho_0\sqrt{s+\beta})K_0(\rho\sqrt{s+\beta})](\rho,\rho_0,\xi-\xi_0)S_1(\rho_0,\xi_0)d\xi_0\right)\rho_0 d\rho_0 +$$

$$\int_1^{\rho}\left(\int_0^{\xi}\mathfrak{L}^{-1}[I_0(\rho\sqrt{s+\beta})K_0(\rho_0\sqrt{s+\beta})](\rho,\rho_0,\xi,\xi-\xi_0)S_1(\rho_0,\xi_0)d\xi_0\right)\rho_0 d\rho_0 -$$

$$\int_1^{\infty}\left(\int_0^{\xi}\mathfrak{L}^{-1}\left[\frac{K_0(\rho\sqrt{s+\beta})K_0(\rho_0\sqrt{s+\beta})I_0(\sqrt{s+\beta})}{K_0(\sqrt{s+\beta})}\right](\rho,\rho_0,\xi-\xi_0)S_1(\rho_0,\xi_0)d\xi_0\right)\rho_0 d\rho_0.$$

Having in mind that the expression of $V_1(\rho,\xi)$ is difficult to handle, we opted for solve the problem (11). Using the Laplace transform, the problem leads to,

$$\frac{\partial}{\partial\rho}\left((-\rho)\frac{\partial L[V_H]}{\partial\rho}\right)+\frac{\partial L[V_H]}{\partial\xi}+\rho(s+\beta)L[V_H]=L[\rho S_1(\rho,\xi)] \tag{12}$$

$$lim_{\rho\to\infty}L[V_H](\rho,s)=0$$

$$L[V_H](1,s)=0.$$

The homogeneous equation associated to (14) is a modified Bessel equation of order 0 with general solution:

$$L[V_H](\rho,s)=C_1(s)I_0(\rho\sqrt{(s+\beta)})+C_2(s)K_0(\rho\sqrt{(s+\beta)}).$$

And using the variation of constant method and including the boundary conditions we obtain:

$$C_1(s)=\int_\rho^\infty K_0(x\sqrt{s+\beta})\mathfrak{L}[S_1](x,s)\,x\,dx$$

$$C_2(s)=\int_1^\rho I_0(x\sqrt{s+\beta})\mathfrak{L}[S_1](x,s)\,x\,dx-$$

$$\frac{I_0(\sqrt{s+\beta})}{K_0(\sqrt{s+\beta})}\int_1^\infty K_0(x\sqrt{s+\beta})\mathfrak{L}[S_1](x,s)\,x\,dx.$$

Then the solution is

$$L[V_H](\rho,s)=I_0(\rho\sqrt{(s+\beta)})\int_\rho^\infty K_0(x\sqrt{(s+\beta)})\,L[S_1](x,s)\,x\,dx+$$

$$K_0(\rho\sqrt{s+\beta})\int_1^\rho I_0(x\sqrt{s+\beta})\mathfrak{L}[S_1](x,s)\,x\,dx-$$

$$\frac{K_0(\rho\sqrt{s+\beta})I_0(\sqrt{s+\beta})}{K_0(\sqrt{s+\beta})}\int_1^\infty K_0(x\sqrt{s+\beta})\mathfrak{L}[S_1](x,s)\,x\,dx.$$

Hence, using the convolution theorem,

$$V_H(\rho,s)=\int_\rho^\infty \mathfrak{L}^{-1}[I_0(\rho\sqrt{s+\beta})K_0(x\sqrt{s+\beta})](\rho,x,\xi)*S_1(x,\xi)\,x\,dx+$$

$$\int_1^\rho \mathfrak{L}^{-1}[I_0(x\sqrt{s+\beta})K_0(\rho\sqrt{s+\beta})](\rho,x,\xi)*S_1(x,\xi)\,x\,dx-$$

$$\int_1^\infty \mathfrak{L}^{-1}[\frac{I_0(\sqrt{s+\beta})I_0(x\sqrt{s+\beta})K_0(x\sqrt{s+\beta})}{K_0(\sqrt{s+\beta})}](\rho,x,\xi)*S_1(x,\xi)\,x\,dx=$$

$$\int_1^\infty\left(\int_0^\xi \mathfrak{L}^{-1}[W](\rho,\rho_0,\xi,\xi_0)(\rho_0\,S_1(\rho_0,\xi_0))d\xi_0\right)d\rho_0=V_1(\rho,\xi).$$

This proves that $G(\rho,\rho_0,\xi,\xi_0)=L^{-1}[W](\rho,\rho_0,\xi,\xi_0)$

The calculus of the involved inverse Laplace transform was done in [3]. For example, if $a,b\geq 1$:

$$L^{-1}[I_0(a\sqrt{s+\beta})K_0(\rho\sqrt{s+\beta})](a,b,\xi)=$$

$$\frac{1}{2}\int_\beta^\infty e^{-x\xi}J_0(a\sqrt{x-\beta})J_0(b\sqrt{x-\beta})dx.$$

Then

$$L^{-1}\left[K_0(\rho\sqrt{s+\beta})/K_0(\sqrt{s+\beta})\right] =$$

$$-\frac{1}{\pi}\int_\beta^\infty e^{-x\xi}\,\frac{J_0(\rho\sqrt{x-\beta})Y_0(\sqrt{x-\beta})-J_0(\sqrt{x-\beta})Y_0(\rho\sqrt{x-\beta})}{J_0^2(\sqrt{x-\beta})+Y_0^2(\sqrt{x-\beta})}dx.$$

Then, from the convolution theorem

$$L^{-1}\left[K_0(a\sqrt{s+\beta})K_0(b\sqrt{s+\beta})I_0(\sqrt{s+\beta})/K_0(\sqrt{s+\beta})\right]=$$

$$L^{-1}[I_0(a\sqrt{s+\beta})K_0(\rho\sqrt{s+\beta})]*L^{-1}\left[K_0(\rho\sqrt{s+\beta})/K_0(\rho\sqrt{s+\beta})\right].$$

## III. INCORPORATING THE CONTOUR CONDITION AT ρ=1

In regular Sturm-Lioville problems the Green's function provides a closed form to solve the complete problem, that is, the problem with not null boundary conditions see [5] and [6], but in unbounded domains the method doesn't work because we cannot use the Green's theorem to transform an integral in the domain in an integral in the boundary of the domain. Then we have to use a classical method to solve:

$$\frac{\partial}{\partial\rho}\left((-\rho)\frac{\partial V_C}{\partial\rho}\right)+\frac{\partial V_C}{\partial\xi}+\rho\beta V_C=\rho S_1(\rho,\xi) \tag{13}$$

$$V_C(\rho,0)=0,\ \ \forall\,\rho>1$$
$$\lim\rho\to\infty V_C(\rho,\xi)=0,\ \ \forall\xi>0$$
$$V_C(1,\xi)=f_1(\xi),\ \forall\xi>0.$$

Using the Laplace transform we have

$$\frac{\partial}{\partial\rho}\left((-\rho)\frac{\partial L[V_C]}{\partial\rho}\right)+\frac{\partial L[V_C]}{\partial\xi}+\rho(s+\beta)L[V_C]=\rho L[S_1(\rho,\xi)] \tag{14}$$

$$lim_{\rho\to\infty}\mathrm{L}[V_C](\rho,s)=0,$$

$$\mathrm{L}[V_C](1,s)=\mathrm{L}[f_1](s).$$

The homogeneous equation associated to (14) is a modified Bessel equation of order 0 with general solution:

$$\mathrm{L}[V](\rho,s)=C_1(s)I_0(\rho\sqrt{s+\beta})+C_2(s)K_0(\rho\sqrt{s+\beta})$$

And using the variation of constant method and including the boundary conditions we obtain that

$$L[V](\rho,s)=I_0(\rho\sqrt{s+\beta})\int_\rho^\infty K_0(x\sqrt{s+\beta})L[S_1](\rho,s)\,x\,dx+$$

$$K_0(\rho\sqrt{s+\beta})\int_1^\rho I_0(x\sqrt{s+\beta})L[S_1](\rho,s)\,x\,dx-$$

$$K_0(\rho\sqrt{s+\beta})\,I_0(\sqrt{s+\beta})/K_0(\sqrt{s+\beta})\int_1^\infty K_0(x\sqrt{s+\beta})L[S_1](\rho,s)\,x\,dx+$$

$$K_0(\rho\sqrt{s+\beta})/K_0(\sqrt{s+\beta})\,\mathrm{L}[f_1](s).$$

Hence

$$V_C(\rho,\xi) = V(\rho,\xi) - \mathfrak{L}^{-1}[\frac{K_0(\rho\sqrt{s+\beta})}{K_0(\sqrt{s+\beta})}](\rho,\xi) * f_1(\xi) =$$

$$\int_1^{\infty}( \int_0^{\xi} G(\rho,\rho_0,\xi,\xi_0)\, S_1(\rho_0,\xi_0)\, d\xi_0)\, \rho_0\, d\rho_0 +$$

$$L^{-1}[K_0(\rho\sqrt{s+\beta}) / K_0(\sqrt{s+\beta})] * f_1(\xi)$$

We recall that the Green's function doesn't plays any role in the incorporation to the solution the contour condition at $\rho$= 1. And also we recall that if $f_1(\xi) = B$, where

$$B = \frac{\sigma k(T_b - T_C)}{j_0^2 r_0^2}$$

this solution coincides with the dimensionless distribution of temperatures in the tissue during RF ablation with needle like internally cooled electrode obtained in [3].

## IV. INCORPORATING THE INITIAL CONDITION

We will solve the following problem

$$\frac{\partial}{\partial\rho}\left((-\rho)\frac{\partial V_I}{\partial\rho}\right) + \frac{\partial V_I}{\partial\xi} + \rho\beta V_I = \rho S_1(\rho,\xi) \tag{15}$$

$$V_I(\rho,0) = g_1(\rho),\ \forall\ \rho > 1$$

$$lim_{\rho\to\infty} V_I(\rho,\xi) = 0,\ \forall\xi > 0$$

$$V_I(1,\xi) = f_1(\xi), \forall\xi > 0.$$

Using the Laplace transform we have

$$\frac{\partial}{\partial\rho}\left((-\rho)\frac{\partial L[V_I]}{\partial\rho}\right) + \frac{\partial L[V_I]}{\partial\xi} + \rho(s+\beta)L[V_I] = \rho(S_1(\rho,\xi) - g_1(\rho)) \tag{16}$$

$$lim_{\rho\to\infty} L[V_I(\rho,\xi)] = 0,$$

$$L[V_I](1,s) = L[f_1](s).$$

Hence, as $L^{-1}[g_1(\rho)] = g_1(\rho)\delta(\xi)$,

$$V_I(\rho,\xi) = \int_1^{\infty}\left(\int_0^{\xi} G(\rho,\rho_0,\xi,\xi_0)S_1(\rho_0,\xi_0))d\xi_0\right)\rho_0 d\rho_0 +$$

$$\int_1^{\infty}\left(\int_0^{\xi} G(\rho,\rho_0,\xi,\xi_0)g_1(\rho_0)\delta(\xi_0)d\xi_0\right)\rho_0 d\rho_0 +$$

$$L^{-1}[K_0(\rho\sqrt{s+\beta}) / K_0(\sqrt{s+\beta})] * f_1(\xi) =$$

$$\int_1^{\infty}\left(\int_0^{\xi} G(\rho,\rho_0,\xi,\xi_0)S_1(\rho_0,\xi_0))d\xi_0\right)\rho_0 d\rho_0 +$$

$$\int_1^{\infty} G(\rho,\rho_0,\xi,0)g_1(\rho_0)\ \rho_0\ d\rho_0 +$$

$$L^{-1}[K_0(\rho\sqrt{s+\beta}) / K_0(\sqrt{s+\beta})] * f_1(\xi).$$

We remark that the Green's function plays the expected role in the incorporation of the initial condition to the solution

## V. RESOLUTION OF THE COMPLETE PROBLEM

$$\frac{\partial}{\partial \rho}\left((-\rho)\frac{\partial V}{\partial \rho}\right) + \frac{\partial V}{\partial \xi} + \rho\beta V = \rho S_1(\rho, \xi) \tag{17}$$

$$V(\rho,0) = g_1(\rho), \forall \rho>1$$

$$lim_{\rho\to\infty} V(\rho,\xi) = h_1(\xi), \forall \xi>0$$

$$V(1,\xi) = f_1(\xi), \forall \xi>0.$$

Define $H(\rho,\xi) = V(\rho,\xi) - h_1(\xi)$. Then $H(\rho,\xi)$ satisfies the equation:

$$\frac{\partial}{\partial \rho}\left((-\rho)\frac{\partial H}{\partial \rho}\right) + \frac{\partial H}{\partial \xi} + \rho\beta H = \rho(S_1(\rho,\xi) + h_1'(\xi) + \beta h_1(\xi)) \tag{18}$$

$$H(\rho,0) = g(\rho) - h_1(0), \forall \rho>1$$

$$lim_{\rho\to\infty} H(\rho,\xi) = 0, \forall \xi>0$$

$$H(1,\xi) = f_1(\xi) - h_1(\xi), \forall \xi>0.$$

Then

$$V(\rho,\xi) = \int_1^\infty \left(\int_0^\xi G(\rho,\rho_0,\xi,\xi_0)(S_1(\rho_0,\xi_0) + h_1'(\xi_0) + \beta h_1(\xi_0))d\xi_0\right)\rho_0 d\rho_0 +$$
$$\int_1^\infty \left(\int_0^\xi G(\rho,\rho_0,\xi,\xi_0)(g_1(\rho_0) - h_1(0))\delta(\xi_0)d\xi_0\right)\rho_0 d\rho_0 -$$
$$L^{-1}[K_0(\rho\sqrt{s+\beta}) / K_0(\sqrt{s+\beta})] * (L^{-1}[K_0(\rho\sqrt{s+\beta}) / K_0(\sqrt{s+\beta})] * (h1(\xi) - f_1(\xi)) =$$

$$\int_1^\infty \left(\int_0^\xi G(\rho,\rho_0,\xi,\xi_0)(S_1(\rho_0,\xi_0)d\xi_0\right)\rho_0 d\rho_0 +$$
$$\int_1^\infty G(\rho,\rho_0,\xi,0)g_1(\rho_0)\ \rho_0 d\rho_0 +$$
$$\int_1^\infty \left(\int_0^\xi G(\rho,\rho_0,\xi,\xi_0)(h_1'(\xi_0) + \beta h_1(\xi_0) - h_1(0)\delta(\xi_0))d\xi_0\right)\rho_0 d\rho_0 -$$
$$L^{-1}[K_0(\rho\sqrt{s+\beta}) / K_0(\sqrt{s+\beta})] * (L^{-1}[K_0(\rho\sqrt{s+\beta}) / K_0(\sqrt{s+\beta})] * (h_1(\xi) - f_1(\xi))$$

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