Kinematic Surface Generated by an Equiform Motion of Astroid Curve

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Abstract**—** *In this paper, a kinematic surface using equiform motion of an astroid curve in Euclidean 3-space E^3 is generated. The main results given in this paper: the surface foliated by equiform motion of astroid curve has a constant Gaussian and mean curvatures if motion of astroids is in parallel planes. Also, the geodesic curves on this surface are obtained. Additionally, special Weingarten of such surface is investigated. Finally, for some special cases new examples are constructed and plotted.*

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I. INTRODUCTION

The kinematic geometry is dedicated to the study of geometrical and temporal characteristics of movement procedures, mechanical aspects such as masses, forces, and so on remain unconsidered. With additionally neglect of the temporal aspect, one can speak more exactly of kinematic geometry. Regarding the relations to the mechanics and to technical applications in mechanical engineering, the kinetics were turned originally to the movements of rigid systems in the Euclidean plane and the three-dimensional Euclidean space and arrived here in the second half of the 19 Century and at the beginning of the last century too more largely.

In recent years interesting applications of kinematics have, for example, been made in areas as diverse as: animal locomotion, biomechanics, geology, robots and manipulators, space mechanics, structural, chemistry and surgery [1].

In [2] R. Lopez proved that cyclic surfaces in Euclidean three-dimensional space with nonzero constant Gaussian curvature are surfaces of revolution. In the case that the Gaussian curvature vanishes on the surface, then the planes containing the circles must be parallel. In [3] Fathi M. Hamdoon studied the corresponding kinematic three-dimensional surface under the hypothesis that its scalar curvature K is constant. In the eighteenth century, Euler proved that the catenoid is the only minimal surface of revolution. In 1860, Riemann found a family of embedded minimal surfaces foliated by circles in parallel planes. Each one of such surfaces is invariant by a group of translations and presents planar ends in a discrete set of heights [4]. At the same time, Enneper proved that in a minimal cyclic surface, the foliating planes must be parallel [5]. As a consequence of Euler, Riemann and Enneper's works, we have that the catenoid and Riemann minimal examples are the only minimal cyclic surfaces in Euclidean space. A century later, Nitsche [6] studied, in 1989, cyclic surfaces with nonzero constant mean curvature and he proved that the only such surfaces are the surfaces of revolution discovered by Delaunay in 1841 [7].Several special motions in equiform planar kinematics have been investigated by [8, 9] and [10]. For more treatment of cyclic surfaces see [11, 12] and [13].

An equiform transformation in the 3-dimensional Euclidean space E^3 is an affine transformation whose linear part is composed of an orthogonal transformation and a homothetical transformation. This motion can be represented by a translation vector d and a rotation matrix A as the following.

$$\bar{x} \to \rho A x + d, \tag{1}$$

where $AA^t = A^t A = I, \quad .2cm\bar{x}, x \in E^3$ and ρ is the scaling factor [3, 14]. An equiform motion is defined if the parameters of (1)- including ρ - are given as functions of a time parameter t. Then a smooth one-parameter equiform motion moves a point x via $\bar{x}(t) = \rho(t) A(t) x(t) + d(t)$. The kinematic corresponding to this transformation group is called an equiform kinematic.

The purpose of this paper is to describe the kinematic surface obtained by the motion of an astroid curve whose Gaussian and mean curvatures K and H are constant, respectively. As a consequence of our result, we prove:

A kinematic 2-dimensional surface obtained by the equiform motion of an astroid curve has a zero Gaussian and mean curvatures if a motion of astroids is in parallel planes. Moreover, using the motion of such surface, the kinematic geometry of geodesic lines is determined. Special Weingarten kinematic surface is studied. Finally, some examples are provided.

II. BASIC CONCEPTS

Here, and in the sequel, we assume that the indices $\{i, j\}$ run over the ranges $\{1,2\}$. The Einstein summation convention will be used, that is, repeated indices, with one upper index and and one lower index, denoted summation over its range. Consider M a surface in E^3 parameterized by

$$X = X(u^i) = X(u,v), \tag{2}$$

and let **N** denote the unit normal vector field on M given by

$$N = \frac{X_1 \wedge X_2}{|X_1 \wedge X_2|}, \qquad X_i = \frac{\partial X}{\partial u^i}, \qquad X_{ij} = \frac{\partial^2 X}{\partial u^i \partial u^j}, \tag{3}$$

where $\wedge$ stands of the cross product of E^3. The metric $\langle \, , \rangle$ in each tangent plane is determined by the first fundamental form

$$I = g_{ij} du^i du^j, \tag{4}$$

with differentiable coefficients

$$g_{ij} = \langle X_i, X_j \rangle. \tag{5}$$

The shape operator of the immersion is represented by the second fundamental form

$$II = -\langle dN, dX \rangle = h_{ij} du^i du^j, \tag{6}$$

with differentiable coefficients

$$h_{ij} = \langle N, X_{ij} \rangle. \tag{7}$$

With the parametrization of the surface (2), the Gaussian and mean curvatures are given by

$$K = Det(h_{ij})/Det(g_{ij}), \tag{8}$$

and

$$H = \frac{1}{2} tr(g^{ij} h_{ij}), \tag{9}$$

respectively, where, (g^{ij}) is the associated contravariant metric tensor field of the covariant metric tenser field (g_{ij}), i.e., $g^{ij} g_{ij} = \delta^i_j$.

The surface M generated by an astroid curve is represented by

$$M : X(u,v) = c(u) + r(u)(\cos^3 v \; t(u) + sin^3 v \; n(u)), \tag{10}$$

where $r(u)$ and $c(u)$ denote the radius and centre of each u-astroid of the foliation, $v \in [0, 2\pi]$. Let $\Phi = \Phi(u)$ be an orthogonal smooth curve to each u-plane of the foliation and represented by its arc length u. We assume that the planes of the foliation are not parallel. Let **t**, **n** and **b** be unit tangent, normal and binormal vectors, respectively, to Φ. Then, Frenet equations of the curve Φ are

$$\begin{pmatrix} t' \\ n' \\ b' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix}, \qquad ' = \frac{d}{du}, \tag{11}$$

where k and τ are the curvature and torsion of $\Phi(u)$, respectively. Observe that $k \neq 0$ because $\Phi(u)$ is not a straight-line. Also, putting

$$c' = \alpha t + \beta n + \gamma b, \tag{12}$$

where α, β, γ are smooth functions in u [2].

III. CONSTANT GAUSSIAN CURVATURE OF M

In this section we will study the constancy of Gaussian curvature K of the surface generated by equiform motion which is locally parameterized by the equation(10).

Putting in $(2.8)\ W = Det(g_{ij})$, we have

$$KW^2 = K_1, \tag{13}$$

where

$$K_1 = [X_1, X_2, X_{11}][X_1, X_2, X_{22}] - [X_1, X_2, X_{12}]^2. \tag{14}$$

Consider now that the surface M is a surface with constant Gauss curvature. After a homothety, it may be assumed, without loss of generality, that the Gaussian curvature is $K = \hat{A}0, 1$ or -1.

3.1 Case *K*=0

Using equation (13) we can express K_1 by trigonometric polynomial on $\cos nv$ and $\sin nv$. Exactly, there exists smooth functions on u, namely A_n and B_n such that (8) writes as

$$0 = KW^2 - K_1 = A_0 + \sum_{i=1}^{16} (A_n \cos iv + B_n \sin iv). \tag{15}$$

Since this is an expression on the independent trigonometric terms $\cos nv$ and $\sin nv$, all coefficients A_n , B_n vanish identically.

After some computations, the coefficients of equation (15) are

$$A_{16} = \frac{-27 r(u)^6 \tau(u)^4}{32768},$$

$$B_{16} = A_{15} = B_{15} = 0.$$

This leads to $\tau(u) = 0$.

Now, the coefficients

$$A_{16} = B_{16} = \cdots A_9 = B_9 = 0,$$

$$A_8 = -\frac{27}{128} r(u)^3 \gamma(u)(4k(u)^2 r(u)\gamma(u) + r'(u)\gamma'(u) - \gamma(u) r''(u)),$$

$$B_8 = -\frac{27}{64} r(u)^3 \gamma(u)(2k(u)\gamma(u) r'(u) + r(u)(\gamma(u) k'(u) - k(u)\gamma'(u))),$$

$$B_8 = -\frac{27}{64} r(u)^3 \gamma(u)(k(u)\beta(u)\gamma(u) - \gamma(u)\alpha'(u) + \alpha(u)\gamma'(u)).$$

The above coefficients equal to zero in the following two cases

(i) $\gamma(u) = 0$, **therefore, we have**

$$A_5 = B_5 = \cdots A_1 = B_1 = A_0 = 0,$$

(ii)

$$4k(u)^2 r(u)\gamma(u) + r'(u)\gamma'(u) - \gamma(u) r''(u) = 0,$$

$$2k(u)\gamma(u) r'(u) + r(u)(\gamma(u) k'(u) - k(u)\gamma'(u)) = 0,$$

$$k(u)\beta(u)\gamma(u) - \gamma(u)\alpha'(u) + \alpha(u)\gamma'(u) = 0,$$

This system of nonlinear ODEs is of second order. Thus, their general solution is much more complicated and can only be solved in special cases. Thus, their solution is also $\gamma(u) = 0$.

Therefore, one can see all coefficients are vanished. Thus we have the proof of the following theorem:

Theorem 3.1 *The surface foliated by equiform motion of astroid curve has a zero Gaussian curvature if a motion of astroid is in parallel planes.*

3.2 Case *K*=1

From (13) similarly as in above case, we have

$$0 = KW^2 - K_1 = A_0 + \sum_{i=1}^{20} (A_n cos\, iv + B_n sin\, iv). \qquad (16)$$

A routine computation of the coefficients yields

$$A_{20} = \frac{81 r(u)^8 \tau(u)^4}{524288},$$

$$B_{20} = A_{19} = B_{19} = 0.$$

By solving the coefficient A_{20}, we have $\tau(u) = 0$. Now, the coefficients

$$A_{20} = B_{20} = A_{19} = B_{19} = 0$$

$$A_{16} = -\frac{81 r(u)^4}{32768} (16k(u)^4 r(u)^4 - 24k(u)^2 r(u)^2 r'(u)^2 + r'(u)^4),$$

$$B_{16} = -\frac{81}{4096} (4k(u)^3 r(u)^7 r'(u) - k(u) r(u)^5 r'(u)^3).$$

By solving system A_{16}, B_{16}, we have $k(u) = 0$, $r(u) = r(constant)$. Now, the coefficients

$$A_{16} = B_{16} = ... = A_{13} = B_{13} = 0,$$

$$A_{12} = -\frac{81}{2048} r^4 (\alpha(u)^4 - 6\alpha(u)^2 \beta(u)^2 + \beta(u)^4),$$

$$B_{12} = -\frac{81}{512} r^4 \alpha(u)\beta(u)(\alpha(u)^2 - \beta(u)^2).$$

This gives

$$\alpha(u) = \beta(u) = 0.$$

Thus we have

$$A_{12} = B_{12} = ... = A_9 = B_9 = 0,$$

$$A_8 = -\frac{81 r(u)^4 \gamma(u)^4}{128},$$

which leads to

$$\gamma(u) = 0.$$

then, one can see all coefficients are vanished. Thus, the conclusion of the above case is: if $K = 1$ or -1 then $r(u) = r$ and $\tau(u) = k(u) = \alpha(u) = \beta(u) = \gamma(u) = 0$. This implies that c is a point $c_0 \in R^3$.

IV. CONSTANT MEAN CURVATURE OF M

In this section, we will study the constancy of the mean curvature H of the surface generated by equiform motion which is locally parameterized by equation(9).

By a manner similar to the previous section 3, we put

$$H^2 = \frac{H_1^2(cos\, nv, sin iv)}{4W^3(cos\, iv, sin iv)}, \tag{17}$$

where

$$H_1 = g_{22}[X_1, X_2, X_{11}] - 2g_{12}[X_1, X_2, X_{12}] + g_{11}[X_1, X_2, X_{22}]. \tag{18}$$

According to (17), we discuss two cases.

4.1 Case *H*=0

Thus one can get

$$H_1^2 = \sum_{i=0}^{13} (A_n \cos iv + B_n \sin iv). \tag{19}$$

Since this is an expression of the independent trigonometric terms $\cos nv$ and $\sin nv$, all coefficients A_n, B_n must vanish identically.

Here, after some computations, the coefficients of the equation (19) are

$$A_{13} = 0,$$

$$B_{13} = \frac{9 r(u)^5 \tau(u)^3}{1024}.$$

In view of the expression of B_{13}, there is one possibility

$$\tau(u) = 0,$$

which leads to the following

$$A_{13} = B_{13} = \cdots = A_9 = B_9 = 0,$$

$$A_8 = \frac{27}{128} r(u)^2 (r(u)r'(u)\gamma'(u) + \gamma(u)(r'(u)^2 - r(u)r''(u))),$$

$$B_8 = \frac{2}{64} r(u)^4 (\gamma(u)k'(u) - k(u)\gamma'(u)).$$

By solving A_8 of $\gamma(u)$, we discuss two cases:

(I) $\gamma(u) = 0$. This leads to all coefficients are vanished.

(II) $\gamma(u) = \frac{C_1 r'(u)}{r(u)} \neq 0$

$$A_8 = 0,$$

$$B_8 = \frac{2}{64} r(u)^4 (k(u)r'(u)^2 + r(u)(k'(u)r'(u) - k(u)r''(u))).$$

By solving B_8 , we have two cases:

(1) $r'(u) = 0$, this is contradiction.

(2) $k(u) = \frac{C_2 r'(u)}{r(u)}$, then, we have

$$B_8 = 0,$$

$$A_7 = -\frac{9}{64} C_1 r(u)(r'(u)(C_2 \beta(u) r'(u) + 3r(u)\alpha'(u)) + \alpha(u)(r'(u)^2 - 3r(u)r''(u))).$$

By solving A_7 , we have two cases:

(a) $C_1 = 0$, this is contradiction.

(b) $\beta(u) = -\frac{1}{C_2 r'(u)^2} (3r(u)r'(u)\alpha'(u) + \alpha(u)(r'(u)^2 - 3r(u)r''(u)))$.

Then,

$$A_7 = 0,$$

$$B_7 = -\frac{9C_1 r(u)}{64 C_2 r'^2} (3r(u)r'(u)(-9r(u)r''(u)\alpha'(u) + 3r(u)r'(u)\alpha''(u) + 5r'(u)^2 \alpha'(u)) + \alpha(u)((C2^2 + 1)r'(u)^4 + 27r(u)^2 r''(u)^2 - 9r(u)^2 r^{(3)}(u)r'(u) - 15r(u)r'(u)^2 r''(u)))$$

$$A_6 = B_6 = 0,$$

$$A_5 = \frac{27}{8} C_1 r(u)^2 (-r'(u)\alpha'(u) + \alpha(u) r''(u)).$$

By solving A_5, we have $\alpha(u) = C_3 r'(u)$,

this leads to

$$B_7 = \frac{9C_1}{64C_2}(1 + C_2^2) C_3 r(u) r'(u)^3.$$

From B_7 we discuss one possibilities, $C_3 = 0$, then,

$$B_7 = A_6 = B_6 = A_5 = B_5 = 0,$$

$$A_4 = \frac{9C_1}{8r(u)}(C_1^2 + (1 + C_2^2) r(u)^2) r'(u)^3,$$

this leads to $r'(u) = 0$, this is contradiction.

From the previous results, we have the proof of the following theorem:

Theorem 4.1 *The surface foliated by equiform motion of astroid curve is a minimal surface if motions of astroid are in parallel planes.*

4.2 Case H^2=1

Thus one can get

$$0 = 4H^2W^3 - H_1^2 = A_0 + \sum_{i=1}^{30}(A_n \cos iv + B_n \sin iv). \tag{20}$$

After some computations, we have

$$A_{30} = \frac{729}{134217728} r(u)^{12} \tau(u)^6.$$

$$B_{30} = 0$$

In view of the expression for A_{30}, there is one possibility

$$\tau(u) = 0.$$

Thus, we obtain

$$A_{30} = B_{30} = \cdots = A_{25} = B_{25} = 0,$$

$$A_{24} = -\frac{729}{2097152} r(u)^6 (64k(u)^6 r(u)^6 - 240k(u)^4 r(u)^4 r'(u)^2 + 60k(u)^2 r(u)^2 r'(u)^4 - r'(u)^6),$$
$$B_{24} = -\frac{729}{524288} r(u)^7 r'(u)(48k(u)^4 r(u)^4 - 40k(u)^2 r(u)^2 r'(u)^2 + 3r'(u)^4)$$

. By solving system A_{24}, B_{24}, This implies that $k(u) = 0,\ r(u) = r,$ which leads to the following

$$A_{24} = B_{24} = \cdots = A_{19} = B_{19} = 0,$$

$$A_{18} = -\frac{729}{32768} r(u)^6 (\alpha(u)^6 - 15\alpha(u)^4 \beta(u)^2 + 15\alpha(u)^2 \beta(u)^4 - \beta(u)^6),$$

In view of the expression of A_{18}, there are six possibilities

$\alpha(u) = \pm\beta(u)$, $\alpha(u) = \pm 2\beta(u) \pm \sqrt{3}\beta(u)$

(a) $\alpha(u) = -2\beta(u) - \sqrt{3}\beta(u).$ Thus we have

$$B_{18} = -\frac{729}{512}(26 + 15\sqrt{3}) r(u)^6 \beta(u)^6,$$

Therefore, $\beta(u) = 0,$

$$B_{18} = A_{17} = \cdots = A_{13} = B_{13} = 0,$$

$$A_{12} = \frac{729}{512} r(u)^6 \gamma(u)^6,$$

Solving A_{12} implies that $\gamma(u) = 0.$

As a consequence, we have the coefficients

$$A_{12} = B_{12} = \cdots = A_1 = B_1 = A_0 = 0.$$

By direct computation, one can see that all remaining possibilities of $\alpha(u)$ conclude the same results.

Remark 4.2 *The surface foliated by equiform motion of astroid curve has nonzero constant mean curvature* $H = 1$ *or* -1 *if* $r(u) = r$ *and* $\tau(u) = k(u) = \alpha(u) = \beta(u) = \gamma(u) = 0$ *which implies that c is a point* $c_0 \in R^3$.

V. GEODESIC CURVES ON M

In this section, we construct and obtain the necessary and sufficient conditions for a curve on the kinematic surface to be a geodesic $(K_g = 0)$. For this purpose, we recall the following definition:

Definition 5.1 *A curve* $\Gamma = \Gamma(u)$ *on* $M \in R^3$ *is a geodesic of M provided its acceleration* Γ'' *is always normal to M [15].*

Making use of the equation (10), the curve $\Gamma(u)$ can be expressed in the form

$$\Gamma(u) = X(u, f(u)), \quad .v = f(u). \tag{21}$$

Since the curve $\Gamma(u)$ is a regular curve on the surface M in E^3, not necessary parameterized by arc length, **N** is the unit normal vector field of the M, the geodesic curvature k_g is given by [16]

$$k_g = \frac{[N, \Gamma'(u), \Gamma''(u)]}{|\Gamma'(u)|^3}. \tag{22}$$

The above equation can be written in the following form

$$k_g \, |\Gamma'(u)|^3 = Q, \tag{23}$$

where $Q = [N, \Gamma'(u), \Gamma''(u)]$, then

$$0 = k_g \,|\,\Gamma'(u)\,|^3 - Q = A_0 + \sum_{i=1}^{12} (A_n \cos iv + B_n \sin iv). \tag{24}$$

Hence, after some computations, the coefficients of the equation (24) are given as

$$A_{12} = 0,$$

$$B_{12} = \frac{3}{2048} r(u)^4 \tau(u)^4 = 0,$$

which implies $\tau(u) = 0$.

Consequently,

$$A_{11} = B_{11} = A_{10} = B_{10} = A_9 = B_9 = 0,$$

$$\begin{aligned} A_8 = -\frac{9}{128} r(u)(k(u)^3 r(u)^2 r'(u) + k(u)^2 r(u)^2 (-11 f'(u) r'(u) + 4 r(u) f''(u)) \\ + k(u)(-4 r(u)^3 f'(u) k'(u) + 7 r(u)^2 f'(u)^2 r'(u) + 2 r'(u)^3 - r(u) r'(u) r''(u) \\ + r'(u)(3 r(u)^2 f'(u)^3 - 2 f'(u) r'(u)^2 + r(u)(k'(u) r'(u) - r'(u) f''(u) + f'(u) r''(u)))) \end{aligned}$$

$$\begin{aligned} B_8 = \frac{9}{64} r(u)^2 (k(u)^4 r(u)^2 - 3k(u)^3 r(u)^2 f'(u) + f'(u) r'(u)(r(u) k'(u) - 2 f'(u) r'(u)) \\ - k(u)^2 (r(u)^2 f'(u)^2 - 2 r'(u)^2 + r(u) r''(u)) + k(u) r(u)(3 r(u) f'(u)^3 \\ + k'(u) r'(u) - 2 r'(u) f''(u) + f'(u) r''(u))), \end{aligned}$$

This system of nonlinear ODEs is of the second order. Since the cases where this system can be explicitly integrated are rare, a numerical solution of the system is in general the only way to compute points on a geodesic. Thus, for simplicity, we consider the two cases

(I) **Case** $r(u) = r$ (constant) : We have

$$A_8 = \frac{9}{32} r^4 k(u)[-f'(u) k'(u) + k(u) f''(u)] = 0,$$

$$B_8 = \frac{9}{64} r^4 k(u)(k(u)^3 - 3k(u)^2 f'(u) - k(u) f'(u)^2 + 3 f'(u)^3) = 0.$$

This implies $k = \pm f'(u)$ or $k = 3 f'(u)$. Therefore, we consider the following cases

(a) $k = 3 f'(u)$: Then we have

$$A_8 = B_8 = A_7 = B_7 = A_6 = B_6 = 0,$$

$$A_5 = \frac{27}{8} r^3 f'(u)[(7\alpha(u) f'(u)^2 + f'(u)\beta'(u) - \beta(u) f''(u)] = 0,$$

$$B_5 = \frac{27}{8} r^3 f'(u)[7\beta(u) f'(u)^2 - f'(u)\alpha'(u) + \alpha(u) f''(u)] = 0,$$

$$\begin{aligned} A_4 = -\frac{9}{8} r^2 [-\alpha(u) f'(u)\alpha(u) - \beta(u) f'(u)\beta'(u) + \alpha(u)^2 f''(u) + \beta(u)^2 f''(u) \\ + 2\gamma(u)(f'(u)\gamma(u) + \gamma(u) f''(u))] = 0 \end{aligned}.$$

Similarly as solving A_8 and B_8 we consider $f'(u) = c$ where c is constant. Thus we have

$$A_5 = \frac{27}{8}c^2r^3(7c\alpha(u) + \beta'(u)) = 0,$$

$$B_5 = \frac{27}{8}c^2r^3(7c\beta(u) - \alpha'(u)) = 0.$$

Thus we obtain

$$\alpha(u) = c_1\cos(7cu) + c_2\sin(7cu),$$

$$\beta(u) = c_2\cos(7cu) - c_1\sin(7cu),$$

where c_1 , c_2 are constants.

From this we obtain

$$A_4 = 9/4cr^2\gamma(u)\gamma'(u) = 0,$$

$$A_4 = 9c^2r^2(-c_1^2 - c_2^2 + 9c^2r^2 + \gamma(u)^2) = 0.$$

This implies $\gamma(u) = \gamma(constant) = \sqrt{9c^2r^2 - c_1^2 - c_2^2}$. Therefore, we obtain

$$A_0 = (12cr(-c_1^2 - c_2^2 + 9c^2r^2)(c_2\cos(7cu - f(u)) - c_1\sin(7cu - f(u)))\sin(2f(u))).$$

This leads to $r = \frac{\sqrt{c_1^2 + c_2^2}}{3c}$, i.e., $\gamma = 0$. Then

$$B_4 = A_3 = B_3 = A_2 = B_2 = A_1 = B_1 = A_0 = 0.$$

(b) $k = \pm f'(u)$, gives the same result as in case (a).

(II) **Case** $k(u) = k$ (constant): The same results as in (I) are obtained.

Now, we give the following theorem

Theorem 5.2 *The geodesic curves on the surface M have the following representations*

$\Gamma(u) = X(u, f(u))$,

M: $X(u,v) = c(u) + r(u)(\cos^3 v\ t(u) + \sin^3 v\ n(u))$,

$c(u) = \{\xi_1 t(u) + \xi_2 n(u)\}$,

where $v = f(u) = cu + h$,

$$\xi_1 = \frac{1}{27c^3(-1+49c^2)}\Big((1-49c^2) f\cos(u) - 9c^2(1+21c^2)b\cos(7cu) - e\sin(u) + 49c^2 e\sin(u) + 9c^2 a\sin(7cu) + 189c^4 a\sin(7cu) \Big),$$

$$\xi_2 = \left(\frac{(-1+49c^2)e\cos(u) + 90c^3 a\cos(7cu) - f\sin(u) + 49c^2 f\sin(u) + 90c^3 b\sin(7cu)}{9c^2(-1+49c^2)}\right),$$

$t(u) = \{-Sin(3cu), Cos(3cu), 0\},$

$n(u) = \{-Cos(3cu), -Sin(3cu), 0\},$

a, b, c, f and h are constants.

VI. SW-SURFACE

In this section, we construct and obtain the necessary and sufficient conditions for a surface M to be a special Weingarten surface. For this purpose, we recall the following definition:

Definition 6.1 *A surface M in Euclidean 3-space* R^3 *is called a special Weingarten surface if there is relation between its Gaussian and mean curvatures such that* $U(K,H)=0$, *and we abbreviate it by SW-surface [17].*

We can express this as the following condition:

$$aH + bK = c, \tag{25}$$

where a, b and c are constants and $a^2 + b^2 \neq 0$. We can rewrite The Gaussian and mean curvatures of a surface M as the following forms By using the equations (13) and (17),without loss of generality we can take $a=1$, the condition (25), can be written in the following form

$$\frac{H_1}{2W^{3/2}} + b\frac{K_1}{W^2} = c, \tag{26}$$

or, equivalently

$$H_1 W^{1/2} = 2(cW^2 - bK_1). \tag{27}$$

Squaring both sides, we have

$$H_1^2 W - 4(cW^2 - bK_1)^2 = 0. \tag{28}$$

6.1 Case *c*=2

In this case, we discuss the equation (28) at $c=0$, thus it become as a form

$$H_1^2 W - 4(bK_1)^2 = 0 \tag{29}$$

By using equation (14), (18) and a manner similar to the previous sections, we can express (29) as the form

$$\sum_{i=0}^{36}(A_n \cos(iv) + B_n \sin(iv)) = 0. \tag{30}$$

After some computations, the coefficients of equation (30) are

$$A_{36} = -\frac{729a^2 r(u)^{14}\tau(u)^8}{2147483648},$$

$$B_{36} = A_{35} = B_{35} = 0.$$

This gives us one possibility, $\tau(u)=0$. Then

$$A_{36} = B_{36} = \dots B_{25} = 0,$$

$$A_{24}=\frac{6561a^2}{8388608}r(u)^6(-32k(u)^3r(u)^6\gamma(u)k'(u)\gamma'(u)+16k(u)^4r(u)^6\gamma(u)^2-8k(u)r(u)^3\gamma(u)k'(u)r'(u)$$
$$(-3r(u)r'(u)\gamma'(u)-2\gamma(u)(r'(u)^2-r(u)r''(u)))+r'(u)^2(-4r(u)^4\gamma(u)^2k'(u)^2+\gamma(u)^2r'(u)^4$$
$$+2r(u)\gamma(u)r'(u)^2(r'(u)\gamma'(u)-\gamma(u)r''(u))+r(u)^2(r'(u)\gamma'(u)-\gamma(u)r''(u))^2)$$
$$+4k(u)^2r(u)^2(4r(u)^4\gamma(u)^2k'(u)^2-\gamma(u)^2r'(u)^4+2r(u)\gamma(u)r'(u)^2(-3r'(u)\gamma'(u)$$
$$+\gamma(u)r''(u))-r(u)^2(6r'(u)^2\gamma'(u)^2-6\gamma(u)r'(u)\gamma'(u)r''(u)+\gamma(u)^2r''(u)^2))).$$

Here, we discuss two possibilities

(i) $\gamma(u)=0$. Then, one can see that all coefficients are vanished.

(ii) $\gamma(u)=\dfrac{C_1\sqrt{4k(u)^2r(u)^2-4k(u)r(u)r'(u)-r'(u)^2}}{r(u)}\neq 0$, then, we have

$$B_{24}=\frac{6561a^2C_1^2r(u)^6(4k(u)^2r(u)^2+r'(u)^2)^2(k(u)r'(u)^2+r(u)(k'(u)r'(u)-k(u)r''(u)))^2}{(1048576(4k(u)^2r(u)^2-4k(u)r(u)r'(u)-r'(u)^2))},$$

also, this gives us two possibilities

(a) $(4k(u)^2r(u)^2+r'(u)^2)=0$, i.e. $k(u)=0,\ r'(u)=0,$

this implies $\gamma(u)=0$, this is contradiction.

(b) $k(u)r'(u)^2+r(u)(k'(u)r'(u)-k(u)r''(u))=0.$

By solving above deferential equation we get $k(u)=\dfrac{C_2r'(u)}{r(u)}$, then

$$A_{22}=-\frac{729a^2C_1^2(-1-4C_2+4C_2^2)}{2097152}r(u)^4r'(u)^2(9r(u)^2r'(u)^2((-1+4C_2^2)\alpha'(u)^2+8C_2\alpha'(u)\beta'(u)$$
$$+(1-4C2^2)\beta'(u)^2)+6r(u)\beta(u)r'(u)(r'(u)^2(C_2(3+4C_2^2)\alpha'(u)+\beta'(u))+3r(u)(-4C_2\alpha(u)$$
$$+(-1+4C_2^2)\beta(u))r''(u))-\alpha(u)^2((1+3C_2^2+4C_2^4)r'(u)^4-6r(u)r'(u)]^2r''(u)+9(1-4C_2^2)$$
$$r(u)^2r''(u)^2)+\beta(u)^2((1+3C_2^2+4C2^4)r'(u)^4-6r(u)r'(u)^2r''(u)+9(1-4C_2^2)r(u)^2r''(u)^2)$$
$$+2\alpha(u)(2C2\beta(u)(r'(u)^2-6r(u)r''(u))((1+2C_2^2)r'(u)^2-3r(u)r''(u))-3r(u)r'(u)(r'(u)^2(\alpha(u)$$
$$-C_2(3+4C_2^2)\beta'(u))+3r(u)((-1+4C_2^2)\alpha'(u)+4C_2\beta'(u))r''(u)))).$$

The solution of this deferential equation is very difficult, thus $(-1-4C_2+4C_2^2)=0$ i.e., $C_2=\dfrac{1}{2}\pm\dfrac{1}{\sqrt{2}}$ or $r'(u)=0$, in two cases $\gamma(u)=0$, thus, this is contradiction.

This leads to the following theorem:

Theorem 6.2 *The kinematic surface generated by an equiform motion of astroid curve is a special Weingarten surface with condition* $H+bK=0$ *if and only if motion of astroid is in parallel planes.*

6.2 Case *c*≠0

By the same way in above subsection, we can express(28) as the following form

$$\sum_{i=0}^{40}(A_n \cos(iv) + B_n \sin(iv)) = 0. \tag{31}$$

After some computations, the coefficients of equation (31) are

$$A_{40} = -\frac{6561c^2 r(u)^1 6\tau(u)^8}{137438953472}$$

$$B_{40} = A_{39} = B_{39} = 0.$$

This gives us one possibility $\tau(u) = 0$ Then

$$A_{32} = -\frac{6561c^2}{536870912} r(u)^8 (256k(u)^8 r(u)^8 - 1792k(u)^6 r(u)^6 r'(u)^2 + 1120k(u)^4 r(u)^4 r'(u)^4$$

$$-112k(u)^2 r(u)^2 r'(u)^6 + r'(u)^8),$$

$$B_{32} = -\frac{6561c^2}{33554432} k(u) r(u)^9 r'(u)(64k(u)^6 r(u)^6 - 112k(u)^4 r(u)^4 r'(u)^2 + 28k(u)^2 r(u)^2 r'(u)^4 - r'(u)^6),$$

By solving these two deferential equations, we obtained $r(u) = r(constant), \quad k(u) = 0,$

$$A_{24} = -\frac{6561c^2}{2097152} r^8 (\alpha(u)^8 - 28\alpha(u)^6 \beta(u)^2 + 70\alpha(u)^4 \beta(u)^4 - 28\alpha(u)^2 \beta(u)^6 + \beta(u)^8),$$

$$B_{24} = \frac{6561c^2}{262144} r^8 \alpha(u)\beta(u)(\alpha(u)^6 - 7\alpha(u)^4 \beta(u)^2 + 7\alpha(u)^2 \beta(u)^4 - \beta(u)^6).$$

By solving these two equations, we obtained $\alpha(u) = 0, \quad \beta(u) = 0,$ then

$$A_{16} = -\frac{6561c^2 r^8 \gamma(u)^8)}{8192}.$$

Thus, $\gamma(u) = 0$, and therefore, one can see all coefficients are vanished.

Remark 6.3 *The kinematic surface generated by an equiform motion of astroid curve is a special Weingarten surface with condition* $H + bK = c$ *if and only if it has nonzero constant Gaussian and mean curvatures.*

VII. EXAMPLES

In this section to illustrate our investigation, we give two examples:

Example 1 (zero Gaussian curvature) : Consider the circle Φ given by

$$\Phi(u) = \{cos(u), sin(u), 0\}.$$

Using (12), (10)and after some computations, we have

$$c(u) = \{-cos(6u), sin(6u), 0\}.$$

Therefore, the representation of the surface generated by the astroid curve is

$$X(u,v) = \{-cos(6u) - r(cos(u)\,cos(v)^3 + sin(u)\,sin(v)^3),$$

$$-r\,cos(v)^3\,sin(u) + sin(6u) + r\,cos(u)\,sin(v)^3, 0\}$$

Thus, Fig. 1 displays the surface with zero Gaussian curvature.

For geodesic curves on a surface, we give the following example:

Example 2 (zero geodesic curvature) : Consider the curve Φ given by

$$\Phi(u) = \{3c\cos(3cu), 3c\sin(3cu), 0\}.$$

After some computations, we have the representation of the surface generated by the astroid curve as

$$X(u,v) = \{\frac{1}{27c^3}(-\frac{9}{4}c^2\sqrt{a^2+b^2}(\sin(3cu-3v)+3\sin(3cu+v))-\sin(3cu)\zeta_1-3c\cos(3cu)\zeta_2),$$
$$\frac{1}{27c^3}(\frac{9}{4}c^2\sqrt{a^2+b^2}(\cos(3cu-3v)+3\cos(3cu+v))+\cos(3cu)\zeta_1-3c\sin(3cu)\zeta_2),0\}$$

$$\Gamma(u) = \{\frac{1}{27c^3}(-\frac{9}{4}c^2\sqrt{a^2+b^2}(3\sin(4cu+h)+\sin(3cu-3(cu+h)))-\sin(3cu)\zeta_1-3c\cos(3cu)\zeta_2),$$
$$\frac{1}{27c^3}(\frac{9}{4}c^2\sqrt{a^2+b^2}(3\cos(4cu+h)+\cos(3cu-3(cu+h)))+\cos(3cu)\zeta_1-3c\sin(3cu)\zeta_2),0\},$$

$$\zeta_1 = \frac{1}{49c^2-1}\left(189ac^4\sin(7cu)+9ac^2\sin(7cu)-9b\left(21c^2+1\right)c^2\cos(7cu)\right.$$
$$\left.+49c^2e\sin(u)+\left(f-49c^2f\right)\cos(u)-e\sin(u)\right),$$

$$\zeta_2 = \frac{\left(90ac^3\cos(7cu)+90bc^3\sin(7cu)+\left(49c^2-1\right)e\cos(u)+49c^2f\sin(u)-f\sin(u)\right)}{49c^2-1}.$$

The geodesic curves are shown in Figs. $(2-4)$.

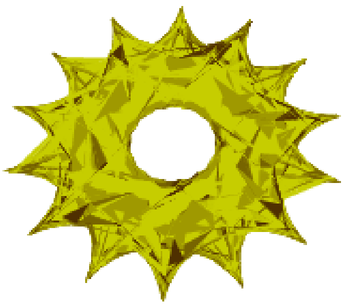

FIGURE 1: THE SURFACE M WITH ZERO GAUSSIAN CURVATURE AT $r=5$.

FIGURE 2: THE GEODESIC CURVE $\Gamma(u)$ ON THE SURFACE M AT $a=b=c=e=f=h=1$.

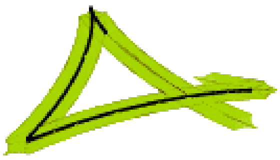

FIGURE 3: THE GEODESIC CURVE $\Gamma(u)$ ON THE SURFACE M AT $a=b=c=e=f=h=0.1$.

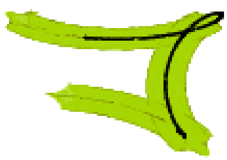

FIGURE 4: THE GEODESIC CURVE $\Gamma(u)$ ON THE SURFACE M AT $a=b=e=f=h=0.1$, $c=0.135$.

VIII. CONCLUSION

In this study, a kinematic surface generated by an equiform motion of astroid curve is considered. Constant Gaussian and mean curvatures of such surface are established. Therefore, the surface foliated by equiform motion of astroid curve has a constant Gaussian and mean curvatures if motion of astroid is in parallel planes. Moreover, the necessary and sufficient conditions for a curve on the kinematic surface M to be a geodesic are given. Using a new technique which is different from that in our papers [18, 19, 20], for SW-surface and it is introduced from a different angle and aspect of [17]. Finally, some examples are given. The field is developing rapidly, and there are a lot of problems to be solved and more work is needed to establish different results of new kinematic surfaces.

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