Stability and Hopf bifurcation for Kaldor-Kalecki model of business cycles with two time delays

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Abstract— *Papers investigate a Kaldor-Kalecki model of business cycle system with two different delays, which described the interaction of the gross product Y and the capital product K. We derived the conditions for the local stability and the existence of Hopf bifurcation at the equilibrium of the system. By applying the normal form theory and center manifold theory, some explicit formulate for determining the stability and the direction of the Hopf bifurcation periodic solutions are obtained. Some numerical simulations by using Mathematica software supported the theoretical results. Finally, main conclusions are given.*

Keywords— Two delays, Hopf bifurcation, Stability, Business cycle, Normal form.

I. INTRODUCTION

According to rational expectation hypothesis, the government will take into account the future capital stock in the process of investment decision. Business cycle, named economic cycle, refers to the total alternate expansion and contraction in economic activities, and these cyclical changes will appear in the form of fluctuation of the comprehensive economic activity indicators such as gross national product, industrial production index, employment and income. In the early research of non-linear business cycle theories, Kaldor [1] assumed that investment depended on gross product and capital stock and proved that, through graph analysis, time-varying nonlinear investment and saving function could result in a business cycle. Chang and Smyth [2] summarized Kaldor's theory on business cycle, established a nonlinear dynamic system which described the gross national product and capital stock changed over time, gave necessary conditions of existence of limit cycles, and provided a rigorous mathematical proof of Kaldor's model proposed in 1940. On this basis, Grasman and Wentzel [3] further improved Kaldor's business cycle model by considering capital loss speed. Along another view, according to the IS-LM model raised by J.R. Hicks and A.H. Hansen, Ackley [4] established a complete Keynes system which reflects the gross domestic product and interest rate changes over time, which is also called standard IS-LM model. Kalecki [5] first considered investment delay in the business cycle model in 1935, and he claimed capital equipment needed a conceived cycle or delay from installation to production. As the theory of delay functional differential equations gradually become more accomplished in 1990s, Krawiec and Szydlowski [6,7] first made a qualitative analysis of the impact of the investment delay on the business cycle. In addition, some other scholars investigated the dynamic properties like stability, many types of bifurcation, existence and stability of periodic solutions in Kaldor–Kalecki model with investment delay [8,9]. The record of business cycle has been kept relatively well during the last 200 years, and business cycle theory, as the core issue of macroeconomics, has been attracting the widespread interests of many economists. The modern business cycle theory can be traced back to the masterpiece of Keynes's theory "The General Theory of Employment, Interest, and Money". Keynes discussed the formation of the business cycle from the perspective of psychological factors based on national income theory. The following system was formulated by Krawiec and Szydlowski [10] who combined two basic models of business cycle: the Kaldor model and the Kalecki model. Kaldor [11] first treated the investment function as a nonlinear (s-shaped) function on Y so that the system may create limit cycles, while Kalecki [12] assumed that the saving part is invested, and that there is a time delay due to the past investment decision. Therefore, the gross product is available in the market after a time lag.

$$\begin{cases} \dot{Y}(t) = \alpha(I(Y,K) - S(Y,K)), \\ \dot{K}(t) = I(Y(t-\tau),K) - \delta K. \end{cases} \tag{1.1}$$

Y is the gross product and K is the capital product of the business cycle; $\alpha > 0$ measures the reaction of the system to the difference between investment and saving; $\delta \in (0,1)$ is the depreciation rate of capital stock; $I, S : R \times R \to R$ are investment and saving functions of Y and K, respectively; τ is a time lag representing the delay for the investment due to the past investment decision. In a business cycle system, delay occurred in the production of investment function not only, also released on capital stock, so to introduce the following model.

$$\begin{cases} \dot{Y}(t) = \alpha(I(Y,K) - S(Y,K)), \\ \dot{K}(t) = I(Y(t-\tau), K(t-\tau)) - \delta K. \end{cases} \tag{1.2}$$

However, for the systems (1.2), the delay that occurs in investment function and happened in the capital stock is not necessarily the same, so we set up two different time delay. By quoting the function of investment and capital saving function in reference [1], $I(Y,K) = I(Y) - \beta K$, $S(Y,K) = \gamma Y$, $\beta > 0$, $\gamma \in (0,1)$. Following model be produced.

$$\begin{cases} \dot{Y}(t) = \alpha(I(Y) - \beta K - \gamma Y), \\ \dot{K}(t) = I(Y(t-\tau_1)) - \beta K(t-\tau_2)) - \delta K. \end{cases} \tag{1.3}$$

The rest of this paper is organized as follows. In Section 2, we analyze the distribution of eigenvalues of the linearized system of (1.3) and derived the conditions for the local stability and the existence of Hopf bifurcation at the equilibrium of the system. In Section 3, by applying the normal form theory and center manifold theory, some explicit formulate for determining the stability and the direction of the Hopf bifurcation periodic solutions are obtained. In Section 4, some numerical simulations by using Mathematica software supported the theoretical results. Finally, main conclusions are given. In Section 5, main conclusions are given.

II. STABILITY AND ANALYSIS OF LOCAL HOPF BIFURCATION

Let $E^* = (Y^*, K^*)$ be an equilibrium point of Sys.(1.3), $I^* = I(Y^*)$, and $u_1 = Y - Y^*$, $u_2 = K - K^*$, $i(s) = I(s + Y^*) - I^*$. Then Sys.(1.3) can be transformed as

$$\begin{cases} \dot{u}_1 = \alpha(i(u_1) - \gamma u_1 - \beta u_2), \\ \dot{u}_2 = i(u_1(t-\tau_1)) - \beta u_2(t-\tau_2)) - \delta u_2. \end{cases} \tag{2.1}$$

Let the Taylor expansion [13] of i at 0 be

$$i(u) = I^{'}(Y^*) + \frac{I^{''}(Y^*)}{2}(u)^2 + o(|u|^3).$$

Then Sys.(2.1) can be transformed as

$$\begin{cases} \dot{u}_1 = \alpha(I^{'}(Y^*)u_1 + \frac{I^{''}(Y^*)}{2}(u_1)^2 - \gamma u_1 - \beta u_2), \\ \dot{u}_2 = I^{'}(Y^*)u_1(t-\tau_1) + \frac{I^{''}(Y^*)}{2}(u_1(t-\tau_1))^2 - \beta u_2(t-\tau_2)) - \delta u_2. \end{cases} \tag{2.2}$$

Then the linear part of Sys.(2.2) at (0,0) becomes

$$\begin{cases} \dot{u}_1 = \alpha(I^{'}(Y^*)u_1 - \gamma u_1 - \beta u_2), \\ \dot{u}_2 = I^{'}(Y^*)u_1(t-\tau_1) - \beta u_2(t-\tau_2)) - \delta u_2. \end{cases} \tag{2.3}$$

And the corresponding characteristic equation is

$$\lambda^2 + (\delta - a)\lambda + (\lambda - a)\beta e^{-\lambda\tau_2} + be^{-\lambda\tau_1} - \delta a = 0, \tag{2.4}$$

where $a = \alpha(I'(Y^*) - \gamma)\lambda, b = \alpha\beta I'(Y^*)$.

Theorem 2.1 When $\tau_1 = \tau_2 = 0$, the characteristic equation of system (2.4) is

$$\lambda^2 + (\delta - a + \beta)\lambda + b - \delta a - \beta a = 0. \tag{2.5}$$

All roots of the equation (2.5)has negative part if and only if

$$(H_{11}) A_{10} = \delta - a + \beta > 0, A_{11} = b - \delta a - \beta a > 0.$$

Then the equilibrium point $E^* = (Y^*, K^*)$ is locally asymptotically stable.

Theorem 2.2 When $\tau_1 > 0, \tau_2 = 0$, assume that (H_{11}) is satisfied. Then (2.2) has a pair of purely imaginary roots $\pm i\omega_{10}$ when $\tau = \tau_{10}$.

Proof.The characteristic equation of system (2.4) is

$$\lambda^2 + (\delta - a + \beta)\lambda + be^{-\lambda\tau_1} - a(\delta + \beta) = 0. \tag{2.6}$$

Clearly, $i\omega_1$ is a root of Eq.(2.6) if and only if $i\omega_1$ satisfies

$$-\omega_1^2 + (\delta - a + \beta)\omega_1 i + b(\cos\omega_1\tau_1 - i\sin\omega_1\tau_1) - a(\beta + \delta) = 0.$$

Separating the real and imaginary parts, it follows

$$\begin{cases} -\omega^2 + b\cos\omega_1\tau_1 = a(\beta + \delta), \\ (\delta - a + \beta)\omega_1 = b\sin\omega_1\tau_1, \end{cases} \tag{2.7}$$

According to conclusions of Beretta and Kuang [14], the stability of the system changes, when the real part of its characteristic root passes through zero point. Therefore, considering the critical situation, let characteristic root λ real part $\mathrm{Re}\,\lambda = 0,$, adding up the squares of both Eq.(2.7), it yields

$$v_1^2 + (c + a^2)v_1 + a^2c - b^2 = 0, \tag{2.8}$$

where $v_1 = \omega_1^2, c = (\delta + \beta)^2$.

Hence Eq.(2.8) has solution v_{10} and v_{11} .Where

$$v_{10} = \frac{-(a^2 + c) + \sqrt{(a^2 + c)^2 - 4(a^2c - b^2)}}{2}, \; v_{11} = \frac{-(a^2 + c) - \sqrt{(a^2 + c)^2 - 4(a^2c - b^2)}}{2}.$$

Assume

$$h(v_1) = v_1^2 + (c + a^2)v_1 + a^2c - b^2 = 0, \tag{2.9}$$

Noting that v_{11} is negative. Eq.(2.9) has one positive root if and only if $(H_{21}) b^2 > a^2c$. The characteristic Eq.(2.6) has a pair of purely imaginary roots $\pm i\omega_{10} = \pm i\sqrt{v_{10}}$. And, hence

$$\tau_{10} = \frac{1}{\omega_{10}} \arccos(\frac{(\omega_{10})^2 + a(\beta + \delta)}{b}).$$

The proof is completed.

Differentiating both sides of (2.6) with respect to τ_1, it follows that

$$(\frac{d\lambda}{d\tau_1})^{-1} = \frac{2\lambda+\delta+\beta-a}{-\lambda be^{-\lambda\tau_1}} - \frac{\tau_1}{\lambda}.$$

That is $$\mathrm{Re}(\frac{d\lambda}{d\tau_1})^{-1}|_{\substack{\lambda=i\omega_{10}\\ \tau_1=\tau_{10}}} = \frac{2\omega^4+((\delta+\beta)^2+a^2)\omega^2}{(\omega b)^2}.$$

Obviously, it is greater than zero. So, the transversality condition is meted. By applying Lemmas 2.1 and 2.2 and condition (H_{21}), we have the following results.

Theorem 2.3 For system (2.2), if the condition is satisfied, the equilibrium E^* of system (2.2) is asymptotically stable for $\tau_1 \in [0,\tau_{10})$. System (2.2) exhibits the Hopf bifurcation at the equilibrium E^* for $\tau_1 = \tau_{10}$.

Theorem 2.4 When $\tau_1 = 0, \tau_2 > 0$, assume that (H_{31}) is satisfied. Then (2.2) has a pair of purely imaginary roots $\pm i\omega_{20}$ when $\tau = \tau_{20}$.

Proof. The characteristic equation of system (2.4) is

$$\lambda^2 + (\delta-a)\lambda + (\lambda-a)\beta e^{-\lambda\tau_2} + b - a\delta = 0. \tag{2.10}$$

Let $i\omega_2(\omega_2>0)$ is a root of (2.10). Then

$$-\omega_2^2 + (\delta-a)\omega_2 i + \beta(i\omega_2\cos\omega_2\tau_2 + ia\sin\omega_2\tau_2 + \omega_2\sin\omega_2\tau_2 - a\cos\omega_2\tau_2) + \beta - a\delta = 0.$$

Separating the real and imaginary parts, it follows

$$\begin{cases} -\omega_2^2 + \beta\omega_2\sin\omega_2\tau_2 - a\beta\cos\omega_2\tau_2 - a\delta + b = 0, \\ (\delta-a)\omega_2 + \beta\omega_2\cos\omega_2\tau_2 + \beta a\sin\omega_2\tau_2, \end{cases} \tag{2.11}$$

Adding up the squares of both Eq.(2.11), it yields

$$\begin{aligned} &v_2^3 + (\delta^2-\beta^2+2a^2-2b)v_2^2 - (2ab\delta - 2a^2(\delta^2-\beta^2) - (a^2-b)^2)v_2 + \\ &a^2b(b-2a\delta) + a^4(\delta^2-\beta^2) = 0, \end{aligned} \tag{2.12}$$

where $\omega_2^2 = v_2$.

From the analysis above, we can obtain that if all parameter values of the system (2.2) are given, the root of the equation (2.12) can be easy to solved by use of the Mathematica software. Therefore, in order to give out the main conclusion in this section, the following assumptions is given.

(H_{31}) The equation (2.12) at least has one negative root.

If the condition (H_{31}) is satisfied. Then, the characteristic Eq.(2.12) has an positive root v_{20} .So, the Eq.(2.10) has a pair of purely imaginary roots $\pm i\omega_{20} = \pm i\sqrt{v_{20}}$. And, hence

$$\tau_{20} = \frac{1}{\omega_{20}}\arccos(\frac{ab-\delta(a^2+\omega_{20}^2)}{\beta(a^2+\omega_{20}^2)}).$$

The proof is completed.

Differentiating both sides of (2.10) with respect to τ_2, it follows that

$$(\frac{d\lambda}{d\tau_2})^{-1} = \frac{2\lambda+\delta+\beta e^{-\lambda\tau_2}-a}{\beta\lambda(\lambda-a)e^{-\lambda\tau_2}} - \frac{\tau_2}{\lambda}.$$

That is $\mathrm{Re}(\frac{d\lambda}{d\tau_2})^{-1}|_{\tau_2=\tau_{20}}^{\lambda=i\omega_{20}} = \dfrac{2\omega_{20}^6+(\delta^2+\beta^2+3a^2)\omega_{20}^4+a^2(\delta^2+\beta^2+a^2)\omega_{20}^2}{(\beta^2\omega^4+(a\beta\delta)^2)(a^2+\omega^2)}.$

Obviously, it is greater than zero. So the transversality condition is meted. By applying Lemmas 2.4 and condition (H_{31}), we have the following results.

Theorem 2.5 For system (2.2), if the condition (H_{31}) is satisfied, the equilibrium E^* of system (2.2) is asymptotically stable for $\tau_2 \in [0,\tau_{20})$. System (2.2) exhibits the Hopf bifurcation at the equilibrium E^* for $\tau_2=\tau_{20}$.

Theorem 2.6 When $\tau_1=\tau_2=\tau$, assume that (H_{41}) is satisfied. Then (2.2) has a pair of purely imaginary roots $\pm i\omega_0$ when $\tau=\tau_0$.

Proof. The characteristic equation of system (2.4) is

$$\lambda^2+(\delta-a)\lambda+((\lambda-a)\beta+b)e^{-\lambda\tau}-a\delta=0. \tag{2.13}$$

Let $i\omega(\omega>0)$ is a root of (2.13). Then

$$-\omega_2^2+(\delta-a+\beta e^{-\lambda\tau})\omega i-(a\beta-b)e^{-\lambda\tau}-a\delta=0.$$

Separating the real and imaginary parts, it follows

$$\begin{cases} -\omega^2+\beta\omega\sin\omega\tau-(a\beta-b)\cos\omega\tau-a\delta=0, \\ (\delta-a)\omega+\beta\omega\cos\omega\tau+(\beta a-b)\sin\omega_2\tau_2=0, \end{cases} \tag{2.14}$$

Adding up the squares of both Eq.(2.14), it yields

$$v^3+(\delta^2-\beta^2+a^2+(\frac{b}{\beta}-a)^2)v^2+(2(\frac{b}{\beta}-\delta)(\frac{b}{\beta}-a)a\beta-2(a\beta-b)^2+$$

$$(a(a-\frac{b}{\beta})+\delta\frac{b}{\beta})^2)v+(a-\frac{b}{\beta})^2((a\beta-b)^2-(a\delta)^2)=0, \tag{2.15}$$

where $\omega^2=v$.

From the analysis above, we can obtain that if all parameter values of the system (2.2) are given, the root of the equation (2.15) can be easy to solved by use of the Mathematica software. Therefore, in order to give out the main conclusion in this section, the following assumptions is given.

(H_{41}) The equation (2.15) at least has one negative root.

If the condition (H_{41}) is satisfied. Then, the characteristic Eq.(2.15) has an positive root v_0. So, Eq.(2.13) has a pair of purely imaginary roots $\pm i\omega_0=\pm i\sqrt{v_0}$. And, hence

$$\tau_0=\frac{1}{\omega_0}\arccos(\frac{\omega_0^2(b-\delta\beta)+a\delta(b-a\beta)}{\beta^2\omega_0^2+(a\beta-b)^2}).$$

The proof is completed.

Differentiating both sides of (2.13) with respect to τ , it follows that

$$(\frac{d\lambda}{d\tau})^{-1}=\frac{2\lambda+\delta+\beta e^{-\lambda\tau}-a}{\lambda(\beta(\lambda-a)+b)e^{-\lambda\tau}}-\frac{\tau}{\lambda}.$$

That is

$$\mathrm{Re}(\frac{d\lambda}{d\tau})^{-1}|_{\tau=\tau_0}^{\lambda=i\omega_0}=\frac{2\beta^2\omega_0^6+(2(b-a\beta)^2+\beta^2(a^2+\delta^2))\omega_0^4+(b-a\beta)^2(\delta^2+a^2)\omega_0^2}{(\beta^2\omega_0^2+(a\beta-b)^2)^2}.$$

Obviously, it is greater than zero. The transversality condition is meted. By applying Theorem 2.5 and condition (H_{41}), we havethe following results.

Theorem 2.7 For system (2.2), if the condition (H_{32}) is satisfied, the equilibrium E^* of system (2.2) is asymptotically stable for $\tau\in[0,\tau_0)$. System (2.2) exhibits the Hopf bifurcation at the equilibrium E^* for $\tau=\tau_0$.

Theorem 2.8 When $\tau_1>0,\tau_2\in(0,\tau_{20})$, assume that (H_{51}) is satisfied. Then (2.2) has a pair of purely imaginary roots $\pm i\omega_1^*$ when $\tau=\tau_1^*$.

Proof. Let $i\omega_1^{'}(\omega_1^{'}>0)$ is a root of (2.4). Then

$$-(\omega_1^{'})^2+(\delta-a)\omega_1^{'}i+\beta(i\omega_1^{'}-a)(\cos\omega_1^{'}\tau_2-i\sin\omega_1^{'}\tau_2)+b(\cos\omega_1^{'}\tau_1-i\sin\omega_1^{'}\tau_1)-a\delta=0.$$

Separating the real and imaginary parts, it follows

$$\begin{cases}-(\omega_1^{'})^2+\beta\omega_1^{'}\sin\omega_1^{'}\tau_2-a\beta\cos\omega_1^{'}\tau_2+b\cos\omega_1^{'}\tau_1-a\delta=0,\\(\delta-a)\omega_1^{'}+\beta\omega_1^{'}\cos\omega_1^{'}\tau_2+\beta a\sin\omega_1^{'}\tau_2-b\sin\omega_1^{'}\tau_1=0,\end{cases}\tag{2.16}$$

Adding up the squares of both Eq.(2.16), it yields

$$(\omega_1^{'})^4-(2\beta\sin\omega_1^{'}\tau_2)(\omega_1^{'})^3+(\beta^2+\delta^2+a^2+2\beta\delta\cos\omega_1^{'}\tau_2)(\omega_1^{'})^2-\omega_1^{'}2a^2\beta\sin\omega_1^{'}\tau_2$$
$$+a^2(\beta^2+\delta^2)+2a^2\beta\delta\cos\omega_1^{'}\tau_2-b^2=0.\tag{2.17}$$

From the analysis above, we can obtain that if all parameter values of the system (2.2) are given, the root of the equation (2.17) can be easy to solved by use of the Mathematica software. Therefore, in order to give out the main conclusion in this section, the following assumptions is given.

(H_{51}) The equation (2.17) at least has one negative root.

If the condition (H_{51}) is satisfied. So, the characteristic Eq.(2.17) has an positive root $i\omega_1^*$. Eq.(2.4) has a pair of purely imaginary roots $\pm i\omega_1^*$. And, hence

$$\tau_1^*=\frac{1}{\omega_1^*}\arccos(\frac{(2a-\delta)(\omega_1^*)^2+a^2\delta-ab\cos\omega_1^*\tau_2+b\omega_1^*\sin\omega_1^*\tau_2}{(\omega_1^*)^2-a^2}).$$

The proof is completed.

Differentiating both sides of (2.4) with respect to τ_1, it follows that

$$(\frac{d\lambda}{d\tau})^{-1}=\frac{2\lambda+\delta+\beta e^{-\lambda\tau_2}-a-\tau_2\beta(\lambda-a)e^{-\lambda\tau_2}}{\lambda b e^{-\lambda\tau_1}}-\frac{\tau_1}{\lambda}.$$

That is

$$\mathrm{Re}(\frac{d\lambda}{d\tau_1})^{-1}\Big|_{\tau_1=\tau_1^*}^{\lambda=i\omega_1^*}=\frac{2\omega_1^*\cos\omega_1^*\tau_1^*+(\beta-a)\sin\omega_1^*\tau_1^*+\beta(1+a\tau_2)\sin\omega_1^*(\tau_1^*-\tau_2)-\tau_2\beta\omega_1^*\cos\omega_1^*(\tau_1^*-\tau_2)}{\omega_1^* b}.$$

Obviously, $\mathrm{Re}(\frac{d\lambda}{d\tau_1})^{-1}\Big|_{\tau_1=\tau_1^*}^{\lambda=i\omega_1^*}\neq 0$.The transversality condition is meted. By applying Lemmas 2.8 and condition (H_{51}), we have the following results.

Theorem 2.9 For system (2.2), if the condition $\tau_2\in[0,\tau_{20})$ and (H_{51}) is satisfied, the equilibrium E^* of system (2.2) is asymptotically stable for $\tau_1\in[0,\tau_1^*)$. System (2.2) exhibits the Hopf bifurcation at the equilibrium E^* for $\tau_1=\tau_1^*$.

Theorem 2.10 When $\tau_2>0,\tau_1\in[0,\tau_{10})$, assume that (H_{61}) is satisfied. Then (2.2) has a pair of purely imaginary roots $\pm i\omega_2^*$ when $\tau_2=\tau_2^*$.

Proof. Let $\pm i\omega_2^*(\omega_2^*>0)$ is a root of (2.4). Then

$-(\omega_2^{'})^2+(\delta-a)\omega_2^{'}i+\beta(i\omega_2^{'}-a)(\cos\omega_2^{'}\tau_2-i\sin\omega_2^{'}\tau_2)+b(\cos\omega_2^{'}\tau_1-i\sin\omega_2^{'}\tau_1)-a\delta=0.$ Separating the real and imaginary parts, it follows

$$\begin{cases}-(\omega_2^{'})^2+\beta\omega_1^{'}\sin\omega_2^{'}\tau_2-a\beta\cos\omega_2^{'}\tau_2+b\cos\omega_2^{'}\tau_1-a\delta=0,\\(\delta-a)\omega_2^{'}+\beta\omega_2^{'}\cos\omega_2^{'}\tau_2+\beta a\sin\omega_2^{'}\tau_2-b\sin\omega_2^{'}\tau_1=0,\end{cases}\tag{2.18}$$

Adding up the squares of both Eq.(2.18), it yields

$$\begin{aligned}&(\omega_2^{'})^6+(2a^2-\beta^2+\delta^2-2b^2\cos\omega_2^{'}\tau_1)(\omega_2^{'})^4-(2b(a-\delta)\sin\omega_1^{'}\tau_2)(\omega_2^{'})^3+\\&(2a^2(\delta^2-\beta^2)+a^4+b^2-2ab(\delta+a)\cos\omega_2^{'}\tau_1)(\omega_2^{'})^2+2\omega_2^{'}a^2b(a-\delta)\sin\omega_2^{'}\tau_1+\\&a^2(\delta^2-\beta^2+b^2-2ab\delta\cos\omega_2^{'}\tau_1)=0.\end{aligned}\tag{2.19}$$

In order to give out the main conclusion in this section, the following assumptions is given.

(H_{61}) The equation (2.19) at least has one negative root.

If the condition (H_{61}) is satisfied. Then, the characteristic Eq.(2.19) has an positive root $\omega_2^{'}$.Eq.(2.4) has a pair of purely imaginary roots $\pm i\omega_2^{'}$. And, hence

$$\tau_2^*=\frac{1}{\omega_2^*}\arccos(\frac{(2a-\delta)(\omega_1^*)^2+a^2\delta-ab\cos\omega_2^*\tau_1+b\omega_2^*\sin\omega_2^*\tau_1}{(\omega_2^*)^2-a^2}).$$

The proof is completed.

Differentiating both sides of (2.4) with respect to τ_2, it follows that

$$(\frac{d\lambda}{d\tau})^{-1}=\frac{2\lambda+\delta+\beta e^{-\lambda\tau_1}+\beta e^{-\lambda\tau_2}-a-\tau_1 b e^{-\lambda\tau_1}}{\lambda\beta(\lambda-a)e^{-\lambda\tau_2}}-\frac{\tau_2}{\lambda}.$$

That is $\mathrm{Re}(\frac{d\lambda}{d\tau_2})^{-1}|_{\tau_2=\tau_2^*}^{\lambda=i\omega_2^*}=\frac{A}{\beta^2(\omega_2^*)^2((\omega_2^*)^2+a^2)}$, where

$$A=\beta\omega_2^*(a(\delta-a)-2(\omega_2^*)^2)\sin\omega_2^*\tau_2^*-\beta(\omega_2^*)^2(\delta+a)\cos\omega_2^*\tau_2^*+$$
$$\beta\omega_2^*\tau_1(\omega_2^*\cos\omega_2^*(\tau_2^*-\tau_1)+a\sin\omega_2^*(\tau_2^*-\tau_1))-\beta^2(\omega_2^*)^2.$$

Obviously, $\mathrm{Re}(\frac{d\lambda}{d\tau_2})^{-1}|_{\tau_2=\tau_2^*}^{\lambda=i\omega_2^*}\neq 0$. So, the transversality condition is meted. By applying Theorem 2.10 and condition (H_{61}) , we havethe following results.

Theorem 2.11 For system (2.2), if the condition $\tau_1\in[0,\tau_{10})$ and (H_{61}) is satisfied, the equilibrium E^* of system (2.2) is asymptotically stable for $\tau_2\in[0,\tau_2^*)$. System (2.2) exhibits the Hopf bifurcation at the equilibrium E^* for $\tau_2=\tau_2^*$. When $\tau_2>0,\tau_1\in[0,\tau_{10})$, assume that (H_{61}) is satisfied. Then (2.2) has a pair of purely imaginary roots $\pm i\omega_2^*$ when $\tau_2=\tau_2^*$.

III. DIRECTION AND STABILITY OF THE PERIODIC SOLUTIONS

In this section, by applying the normal form and center manifold theory [15,16], we discuss the direction and stability of the bifurcating periodic solutions for $\tau_2>0,\tau_1\in[0,\tau_{10})$. Throughout this section, we always assume that system (1.3) meets the conditions of the Hopf bifurcation. In this section, we assume that $\tau_{1*}<\tau_2^*,\tau_1\in[0,\tau_{10})$. For convenience, let $\tau_2=\mu+\tau_2^*$, clearly, $\mu=0$ is the Hopf bifurcation value of system (1.3). Let t/τ and still denote t.Then the system (1.3) can be rewritten as

$$\dot{u}(t)=L_\mu+F(\mu,u_t), \tag{3.1}$$

where $u(t)=(u_1(t),u_2(t))^T$ and

$$L_\mu(\phi)=(\tau_2^*+\mu)(A'\phi(0)+B'\phi(-\frac{\tau_{1*}}{\tau_2^*})+C'\phi(-1)), \tag{3.2}$$

$$F(\mu,\phi)=(\tau_2^*+\mu)\begin{pmatrix}F_1\\F_2\\F_3\end{pmatrix},$$

where

$$A'=\begin{pmatrix}a & -\alpha\beta\\0 & -\delta\end{pmatrix},\ B'=\begin{pmatrix}0 & 0\\q' & 0\end{pmatrix},\ \text{and}\ C'=\begin{pmatrix}0 & 0\\0 & -\beta\end{pmatrix}.$$

And $q=I'(Y^*),F_1=\alpha q''\phi_1^2(0),F_2=q''\phi_1^2(\frac{-\tau_{1*}}{\tau_2^*}),q''=\frac{I''(Y^*)}{2}$.

By Riesz's representation theorem, there exists bounded variation functions

$$\eta(\theta,\mu):[-1,0]\to R^{2\times 2}\ \text{for}\ \theta\in[-\tau_1^*,0)\ \text{,such that}$$

$$L_\mu\phi=\int_{-1}^0 d\eta(\theta,\mu)\phi(\theta),\phi\in C. \tag{3.3}$$

In fact, we take

$$\eta(\theta,\mu)=\begin{cases}(\tau_2^*+\mu)(A^{'}+B^{'}+C^{'}), & \theta=0,\\ (\tau_2^*+\mu)(B^{'}+C^{'}), & \theta\in[-\frac{\tau_{1*}}{\tau_2^*},0),\\ (\tau_2^*+\mu)C^{'}, & \theta\in(-1,-\frac{\tau_{1*}}{\tau_2^*}),\\ 0, & \theta=-1.\end{cases} \tag{3.4}$$

For $\psi\in C^{'}[-1,0]$, we define that operator A and R is as follows form.

$$A(\mu)\phi(\theta)=\begin{cases}\dfrac{d(\phi(\theta))}{d\theta}, & \theta\in[-1,0),\\ \int_{-1}^{0}d(\eta(\theta,\mu)\phi(\theta)), & \theta=0.\end{cases} \tag{3.5}$$

$$R(\mu)\phi(\theta)=\begin{cases}0, & \theta\in[-1,0),\\ F(\mu,\phi), & \theta=0.\end{cases} \tag{3.6}$$

Then the system (3.1) is equivalent to the following abstract differential equation

$$\dot{u}_t=A(\mu)u_t+R(\mu)u_t, \tag{3.7}$$

where $u_t=(u_1(t+\theta),u_2(t+\theta),u_3(t+\theta))$ for $\theta\in[-\tau_2^0,0]$.

Setting $\psi\in C^{'}[0,\tau]$, we can define an adjoint operator $A^*(0)$ corresponding to $A(0)$ as the following form:

$$A*\psi(s)=\begin{cases}-\dfrac{d\psi(s)}{ds}, & s\in(0,1],\\ \int_{-\tau_2^0}^{0}d(\eta^T(s,0)\psi(-s)), s=0,\end{cases}$$

For $\varphi\in C^{'}([-\tau_2^0,0],R^2)$ and $\psi\in C^{'}[0,\tau_2^0]$, define the bilinear form:

$$<\varphi(s),\phi(\theta)>=\overline{\varphi}(0)\phi(0)-\int_{-1}^{0}\int_{\varepsilon=0}^{\theta}\overline{\varphi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi, \tag{3.8}$$

where $\eta(\theta)=\eta(\theta,0)$. After discussed above, we know that $\pm\tau_2^*\omega_2^*$ is characteristic root of $A(0)$ and A^*. To determine the normal form of operator A, we need to calculate the eigenvectors $\rho(\theta)=(1,\rho_2)^T e^{i\tau_2^*\omega_2^*\theta}$ and $\rho^*(\theta)=D(1,\rho_2^*)^T e^{i\tau_2^*\omega_2^* s}$ of A and A^* corresponding to $+\tau_2^*\omega_2^*$ and $-\tau_2^*\omega_2^*$, respectively. By the definition of the $A(0)$ and A^*. We calculate that

$$\rho_2=\frac{a-i\tau_2^*\omega_2^*}{\alpha\beta},\ \rho_2^*=\frac{-a-i\tau_2^*\omega_2^*}{q^{'}e^{\tau_{2*}\omega_2^*}}.$$

In addition, according to the equation (3.8), there is

$$<\rho^*(s),\rho(\theta)>=\overline{\rho}^*(0)\rho(0)-\int_{-1}^{0}\int_{\xi=0}^{\theta}\overline{\rho}^*(\xi-\theta)d\eta(\theta)q(\xi)d\xi.$$

$$=\overline{D}[1+\rho_2\overline{\rho_2}^*-\int_{-1}^{0}(1,\overline{\rho}_2^*)\theta e^{-i\tau_2^*\omega_2^*\theta}d\eta(\theta)\begin{pmatrix}1\\ \rho_2\end{pmatrix}]$$

$$=\overline{D}(1+\rho_2\overline{\rho}_2^*+\tau_{1*}q^{'}e^{i\omega_2^*\tau_2^*}-\beta\tau_2^*\rho_2\overline{\rho}_2^*e^{-i\omega_2^*\tau_2^*}),$$

Further, let $\langle q^*(s), q(\theta)\rangle = 1$, there is $\overline{D} = (1+\rho_2\overline{\rho}_2^* + \tau_{1*}q' e^{i\omega_2^*\tau_2^*} - \beta\tau_2^*\rho_2\overline{\rho}_2^* e^{-i\omega_2^*\tau_2^*})^{-1}$.

Next, according to the algorithm in paper [17] and the similar calculation process with the paper [18], following parameters expression be getted.

$$g_{20} = 2\tau_2^*\overline{D}(\alpha q'' + \overline{\rho}_2^* q'' e^{-2i\tau_{1*}\omega_2^*}),$$

$$g_{11} = \tau_2^*\overline{D}(2\alpha q'' + \overline{\rho}_2^* q'' e^{i\tau_{1*}\omega_2^*} + e^{-i\tau_{1*}\omega_2^*}),$$

$$g_{02} = 2\tau_2^*\overline{D}(\alpha q'' + \overline{\rho}_2^* q'' e^{i\tau_{1*}\omega_2^*}),$$

$$g_{21} = 2\tau_2^*\overline{D}[\frac{-\alpha}{M}(2W_{11}^{(1)}(0) + \frac{1}{2}W_{20}^{(1)}(0)) - q''\overline{\rho}_2^*\frac{\tau_{1*}}{\tau_2^*}(2W_{11}^{(1)}(0)\rho_2 + W_{20}^{(1)}(0)\overline{\rho}_2)],$$

where

$$\begin{aligned} W_{20}(\theta) &= \frac{ig_{20}}{\tau_2^*\omega_2^*}\rho(0)e^{i\tau_2^*\omega_2^*\theta} + \frac{i\overline{g}_{02}}{3i\tau_2^*\omega_2^*}\overline{\rho}(0)e^{-i\tau_2^*\omega_2^*\theta} + E_1 e^{-2i\tau_2^*\omega_2^*\theta}, \\ W_{11}(\theta) &= -\frac{ig_{11}}{\tau_2^*\omega_2^*}\rho(0)e^{i\tau_2^*\omega_2^*\theta} + \frac{i\overline{g}_{11}}{\tau_2^*\omega_2^*}\overline{\rho}(0)e^{-i\tau_2^*\omega_2^*\theta} + E_2. \end{aligned} \tag{3.9}$$

While, E_1 and E_2 be determined by the following equation.

$$E_1 = 2\begin{pmatrix} 2i\omega_2^* - a & \alpha\beta \\ -q' e^{-2i\omega_2^*\tau_2^*} & 2i\omega_2^* + \delta + \beta e^{-2i\omega_2^*\tau_2^*} \end{pmatrix}^{-1}\begin{pmatrix} \alpha q'' \\ q'' e^{-2i\omega_2^*\tau_2^*} \end{pmatrix},$$

$$E_2 = -\begin{pmatrix} a & -\alpha\beta \\ q' & -(\alpha+\beta) \end{pmatrix}^{-1}\begin{pmatrix} 2\alpha q'' \\ q'' e^{i\tau_{1*}\omega_1^*} + e^{-i\tau_{1*}\omega_1^*} \end{pmatrix}.$$

In the end, we can calculate the following values of the coefficient.

$$\begin{aligned} C_1(0) &= \frac{i}{2\tau_2^*\omega_2^*}(g_{20}g_{11} - 2|g_{11}| - \frac{1}{3}|g_{02}|^2) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_2^*)\}}, \\ \beta_2 &= 2\,\mathrm{Re}\{C_1(0)\}, \\ T_2 &= -\frac{\mathrm{Im}\{C_1(0)\} + \mu_2(\mathrm{Im}\{\lambda'(\tau_2^*)\})}{\tau_2^*\omega_2^*}. \end{aligned}$$

Theorem 3.1

(i) The direction of the Hopf bifurcation is decided by the symbol of μ_2. If $\mu_2 > 0$, then it's supercritical. If $\mu_2 < 0$, it is subcritical.

(ii) If $\beta_2 > 0$ ($\beta_2 < 0$), then the bifurcating periodic solutions is stable (unstable).

(iii) If $T_2 > 0$ ($T_2 < 0$), then the period of the bifurcating periodic solutions increase (decrease).

IV. NUMERICAL SIMULATION

In this section, we shall conduct some numerical simulations to verify the main conclusions of this paper, and discuss the economic meanings of the different dynamic behaviors. In numerical simulations, according to literature [19], we choose the parameter. Let $\gamma = 0.5, \alpha = 2.275, q' = 0.4,\ q'' = -0.088,\ \beta = 0.8, \delta = 0.1,$ then Eq.(2.2) can be written as

$$\begin{cases} \dot{u}_1 = -0.2275u_1 - 1.82u_2 - 0.2202(u_1)^2, \\ \dot{u}_2 = 0.4u_1(t-\tau_1) - 0.8u_2(t-\tau_2) - 0.088(u_1(t-\tau_1))^2 - 0.1u_2. \end{cases} \tag{4.1}$$

With calculation by Mathematica software, when $\tau_1 = \tau_2 = 0$, we get $A_{10} = 1.31992,$

$A_{11} = 1.36142$. So the condition (H_{10}) is satisfied. When $\tau_1 > 0, \tau_2 = 0$, we get $\omega_{10} = 0.62444, \tau_{10} = 2.10366$. If $\tau_1 \in [0, \tau_{10})$, the equilibrium E^* of system (2.2) is asymptotically stable. Once τ_1 more than critical value $\tau_{10} = 2.10366$, the system (2.2) produce a bunch of periodic solutions nearby the equilibrium point E^*. Numerical simulation results are shown in Figs.1 and 2.Similarly, when $\tau_1 = 0, \tau_2 > 0$, we get $\omega_{20} = 1.47391, \tau_{20} = 1.2149$ (Figs.3 and 4).When $\tau_1 = \tau_2 = 0$, we get $\omega_0 = 1.1152, \tau_0 = 0.95603$ (see Figs.5 and 6).When $\tau_1 > 0, \tau_2 = 1.2 \in (0, \tau_{20})$, we get $\omega_1^* = 1.3327, \tau_1^* = 0.2342$ (see Figs.7 and 8). Similarly, when $\tau_2 > 0, \tau_1 = 1.8 \in (0, \tau_{10})$, we get $\omega_2^* = 0.775, \tau_2^* = 0.6286$ (see Figs.9 and 10).

In addition, $\mu_2 = 6.1344 > 0$, $\beta_2 = -121.1632 < 0.$, $T_2 = 11.2663 > 0$ By the theorem (3.1), when $\tau_2 > 0, \tau_1 \in (0, \tau_{10})$, the Hopf bifurcation produced by the system (2.2) is supercritical, and the period of the bifurcating periodic solutions increase.

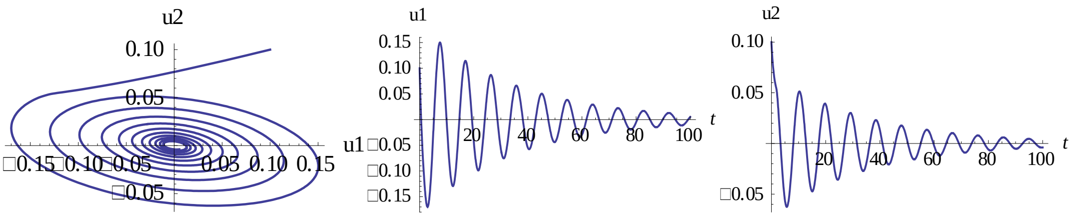

FIGURE 1: THE EQUILIBRIUM E^* IS LOCALLY ASYMPTOTICALLY STABLE WITH $\tau_1 = 1.8 < \tau_{10}$

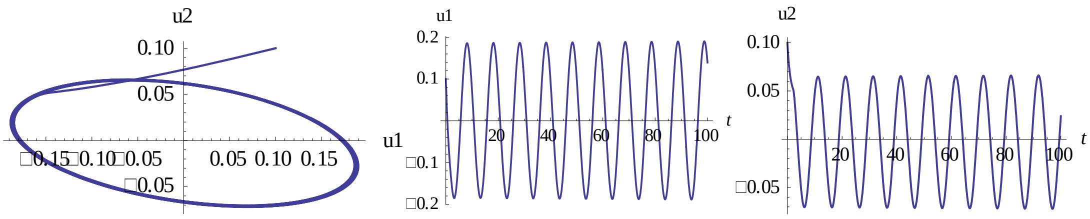

FIGURE 2:THE EQUILIBRIUM E^* BECOMES UNSTABLE AND A HOPF BIFURCATION OCCURS WHEN $\tau_1 = 2.108 > \tau_{10}$

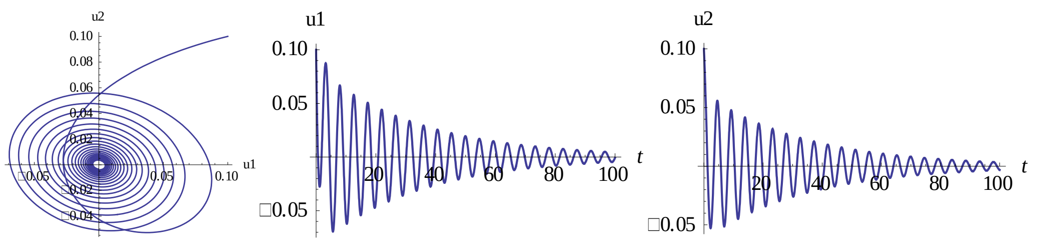

FIGURE 3:THE EQUILIBRIUM E^* IS LOCALLY ASYMPTOTICALLY STABLE WITH $\tau_2 = 1.3 < \tau_{20}$

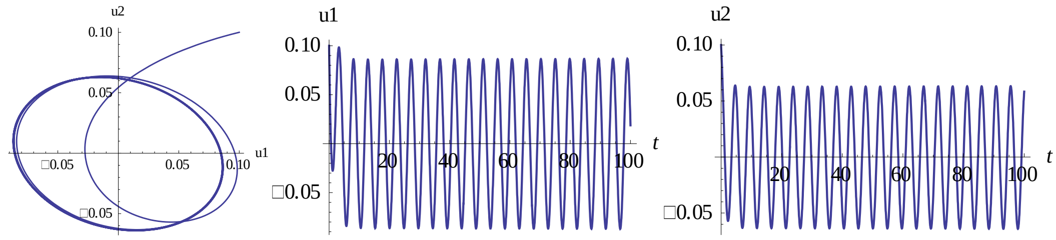

FIGURE 4:THE EQUILIBRIUM E^* **BECOMES UNSTABLE AND A HOPF BIFURCATION OCCURS WHEN** $\tau_2 = 1.381 > \tau_{20}$

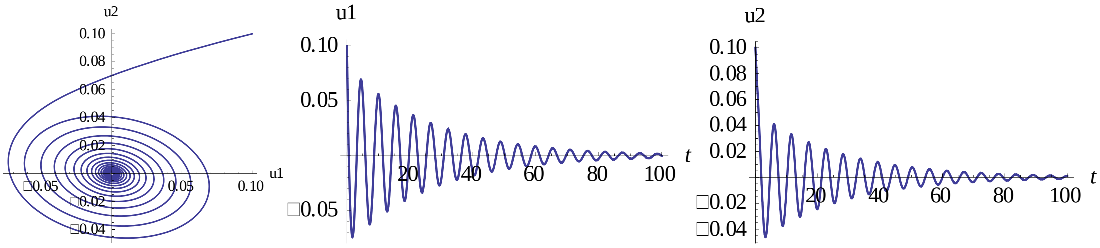

FIGURE 5:THE EQUILIBRIUM E^* **IS LOCALLY ASYMPTOTICALLY STABLE WITH** $\tau_1 = \tau_2 = 0.9 < \tau_{10}$

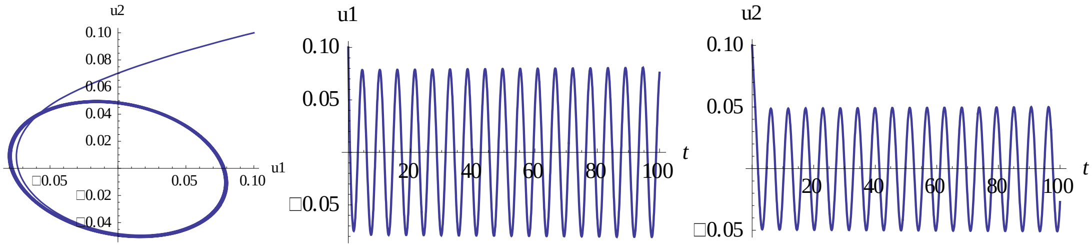

FIGURE 6: THE EQUILIBRIUM E^* **BECOMES UNSTABLE AND A HOPF BIFURCATION OCCURS WHEN** $\tau_1 = \tau_2 = 0.9565 > \tau_{10}$

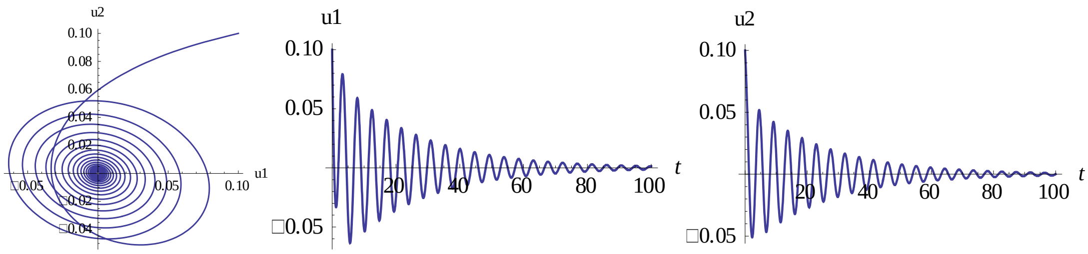

FIGURE 7:THE EQUILIBRIUM E^* **IS LOCALLY ASYMPTOTICALLY STABLE WITH** $\tau_1 = 0.1 < \tau_{1*}, \tau_2 = 1.2$

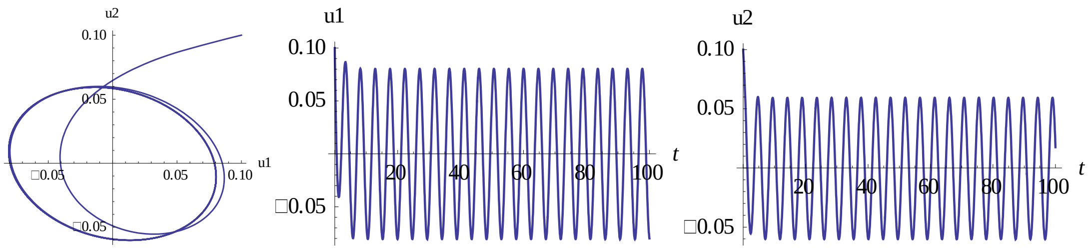

FIGURE 8:THE EQUILIBRIUM E^* **BECOMES UNSTABLE AND A HOPF BIFURCATION OCCURS WHEN**

$$\tau_1 = 0.235 > \tau_{1*}, \tau_2 = 1.2$$

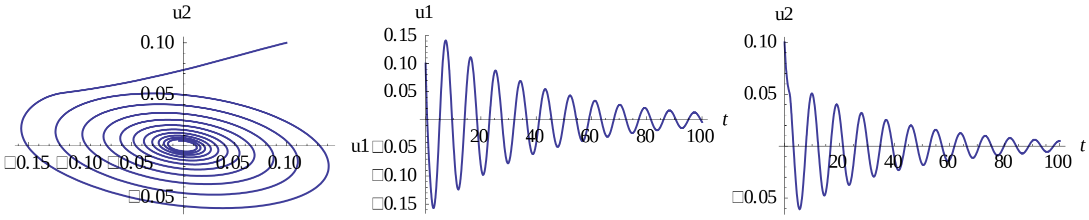

FIGURE 9:THE EQUILIBRIUM E^* **IS LOCALLY ASYMPTOTICALLY STABLE WITH** $\tau_1 = 1.8, \tau_2 = 0.2 < \tau_2^*$

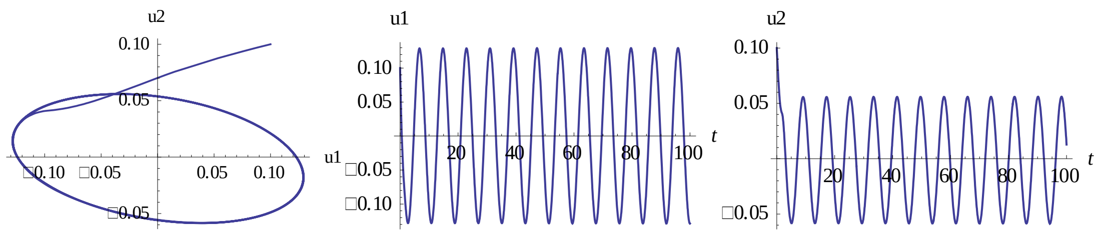

FIGURE 10: THE EQUILIBRIUM E^* BECOMES UNSTABLE AND A HOPF BIFURCATION OCCURS WHEN $\tau_1 = 1.8, \tau_2 = 0.629 > \tau_2^*$

V. CONCLUSION

Since the anticipation of capital stock and its future value are directly interrelated, the government should consider the expectation of capital stock in investment decisions at present stage. At the same time, the implementation of past investment decisions also need a pregnancy period which leads to production delay.

In this paper, the main contribution be lied in the following content: the first, we improved the traditional Kaldor-Kalecki model with two delays in the gross product and the capital stock. We set up the Kaldor-Kalecki model of differential equation with the two delays; The second, we study the stability and Hopf bifurcation. The results indicate that both capital stock and investment lag are the certain factors leading to the occurrence of cyclical fluctuations in the macroeconomic system. Moreover, the level of economic fluctuation can be dampened to some extent if investment decisions are made by the reasonable short-term forecast on capital stock. And finally conduct numerical simulations to prove the conclusions.

The above arguments are well prepared to the further research work, but there are many undeveloped theory that need further explore. The results of this paper can be used as qualitative analysis tool of mathematical economics and business administration.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper.

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AUTHOR CONTRIBUTIONS

Conceived: Xiaohong Wang. Drawing graphics: Xiaohong Wang. Calculated: Xiaohong Wang. Modified: Yanhui Zhai. Wrote the paper: Xiaohong Wang, Ka Long.

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