Study on Some Properties of Anti-centrosymmetric Matrices

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Abstract— *In this paper, the anti-centrosymmetric matrices have been researched. According to the structural characteristics of the anti-centrosymmetric matrix, some new methods have been used to prove the necessary and sufficient conditions of a matrix being anti-centrosymmetric and its properties of eigenvalue and eigenvector; the nonsingularity of the anti-centrosymmetric matrices have been discussed,that the odd order anti-centrosymmetric matrix is singular has been obtained, and two methods of computing inverse of the matrices(even order) have been given.*

Keywords— Anti-centrosymmetric Matrix, Centrosymmetric Matrix, Eigenvalue, Inverse Matrix.

I. INTRODUCTION

Centrosymmetric and anti-centrosymmetric matrices are two kinds of important special matrices, which are widely used in the fields of information theory, numerical analysis, linear system theory and so on. At present, some significant achievements have been acquired from the research on the structure and properties, eigenvalues and inverse of centrosymmetric and anti-centrosymmetric matrices. For example, in reference[3], Tan Ruimei used the definition of anti-centrosymmetric matrix to prove some new conclusions on adjoint matrix, eigenvalue and eigenvector of it; in reference[6], Liu Lianfu discussed the method of computing inverse of anticentrosymmetric matrices in the light of the structure and representation of it. In this paper, based on the previous literatures, some systematic summary research has been made and some new proof methods have been shown for anti-centrosymmetric matrices.

Definition 1. If $A=(a_{ij})\in C^{n\times n}$ and it meets $a_{ij}=a_{n+1-i,n+1-j}, i,j=1,2,\cdots,n$,then A is called centrosymmetric matrix. For centrosymmetric matrix, the elements in line r from the end are precisely those in line r from the beginning which are arranged in the reverse order. The set of the whole n-order centrosymmetric matrices is denoted as $CSR^{n\times n}$.

Definition 2. If $A=(a_{ij})\in C^{n\times n}$ and it meets $a_{ij}=-a_{n+1-i,n+1-j}, i,j=1,2,\cdots,n$,then A is called anti-centrosymmetric matrix. For anti-centrosymmetric matrix, the elements in line r from the end are precisely the opposite number of those in line r from the beginning which are arranged in the reverse order. The set of the whole n-order anti-centrosymmetric matrices is denoted as $ACSR^{n\times n}$.

Definition 3. The matrix $J_n=\begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}=(e_n,\cdots,e_1)$ is called n-order permutation matrix. Obviously, $J_nJ_n=I_n$.

Definition 4. If $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$, then $\hat{A}=\begin{pmatrix} a_{mn} & a_{m,n-1} & \cdots & a_{m1} \\ a_{n-1,n} & a_{n-1,n-1} & \cdots & a_{n-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{1,n-1} & \cdots & a_{11} \end{pmatrix}$ is denoted as the inversion of matrix A.

Lemma 1. $A\in ACSR^{n\times n}$ if and only if $A=-J_nAJ_n$.

Lemma2. If $A\in ACSR^{2n\times 2n}$,then A can be expressed as

$$A=\begin{pmatrix} A_1 & A_2J_n \\ -J_nA_2 & -J_nA_1J_n \end{pmatrix}$$

and if $A\in ACSR^{(2n+1)\times(2n+1)}$, then A can be expressed as

$$A=\begin{pmatrix} A_1 & \alpha & A_2J_n \\ \beta^T & 0 & -\beta^T J_n \\ -J_nA_2 & -J_n\alpha & -J_nA_1J_n \end{pmatrix}$$

among which, A_1, A_2 are n-order square matrices; J_n is n-order permutation matrix; α, β are n dimensional column vectors.

Lemma 3. $A\in ACSR^{n\times n}$ if and only if $A=-\hat{A}$; $A\in CSR^{n\times n}$ as well as $A\in ACSR^{n\times n}$ if and only if $A=O$.

Lemma 4. $A\in ACSR^{n\times n}$ if and only if

(1)When $n=2k, k\in Z^*$, $A=(A_1,-\hat{A}_1)$;

(2))When $n=2k+1, k\in Z^*$, $A=(A_1,\alpha,-\hat{A}_1)$.

among which, α is the column vector of A in line $k+1$, A_1 is constituted by the front k column elements of A with the original order and $\hat{A}_1$ is the inversion of A_1.

Proof: When $n=2k+1$, let $A\in ACSR^{n\times n}=(A_1,\alpha,\hat{A}_2)$ .From Lemma 3,

$$A=(A_1,\alpha,\hat{A}_2)=-\hat{A}=(-\hat{A}_2,-\hat{\alpha},-\hat{A}_1)$$

and from Definition 2 and Definition 4, $a_{ij}=-a_{2k+2-i,2k+2-j}$ and $\alpha=-\hat{\alpha}$, then

$$\begin{array}{ll} a_{11}=-a_{2k+1,2k+1},\cdots,a_{1k}=-a_{2k+1,k+2}, & a_{1,k+2}=-a_{2k+1,k},\cdots,a_{1n}=-a_{2k+1,1} \\ a_{21}=-a_{2k,2k+1},\ \cdots,a_{2k}=-a_{2k,k+2}, & a_{2,k+2}=-a_{2k,k},\ \cdots,a_{2n}=-a_{2k,1} \\ \cdots\cdots & \cdots\cdots \\ a_{2k,1}=-a_{2n},\ \cdots,\ a_{2k,k}=-a_{2,k+2}, & a_{2k,k+2}=-a_{2k},\ \cdots,a_{2k,2k+1}=-a_{21} \\ a_{2k+1,1}=-a_{1n},\ \ldots,\ a_{2k+1,k}=-a_{1,k+2} & a_{2k+1,k+2}=-a_{1k},\cdots,a_{2k+1,2k+1}=-a_{11} \end{array}$$

That is $A_1=-\hat{A}_2$, therefore, $A=(A_1,\alpha,-\hat{A}_1)$.

Similarly, the conclusion(1) can be proved.

II. MAIN CONCLUSIONS

Theorem 1. If $A\in ACSR^{n\times n}$, ξ is the eigenvector of A and λ_0 is the corresponding eigenvalue, then $-\lambda_0$ is also one of the eigenvalues which belong to A, besides, $J_n\xi$ is the eigenvector which is corresponding to $-\lambda_0$.

Proof: From what is known,

$$|\lambda_0 I-A|=0$$

Because of Lemma 1,

$$|J_n(\lambda_0 I)J_n+J_nAJ_n|=0$$

and that is

$$\left|J_n\right|\left|\lambda_0 I + A\right|\left|J_n\right| = 0, \left|\lambda_0 I + A\right| = 0$$

Then

$$\left|(-\lambda)_0 I - A\right| = 0$$

and $-\lambda_0$ is the eigenvalue of A.

Since $A\xi = \lambda_0 \xi$, then $-A\xi = -\lambda_0 \xi$, and according to Lemma 1,

$$J_n A J_n \xi = -\lambda_0 \xi$$

Premultiplying both sides of the equation yields

$$J_n J_n A J_n \xi = (-\lambda_0) J_n \xi$$

From Definition 3,

$$A(J_n \xi) = (-\lambda_0) J_n \xi$$

which shows that $J_n \xi$ is the eigenvector which is corresponding to $-\lambda_0$.

Theorem 2. If $A, B \in ACSR^{n\times n}$, then $kA \in ACSR^{n\times n}, (k \in C)$ and $A + B \in ACSR^{n\times n}$, thus all of the n-order anti-centrosymmetric matrices on the complex field constitute a linear subspace of $C^{n\times n}$.

Theorem 3. The trace of a n-order anti-centrosymmetric matrix is 0, and therefore the sum of the matrix's eigenvalues is 0.

Proof: In terms of Definition 2,

$$a_{ij} = -a_{n+1-i,n+1-j}, i, j = 1, 2, \cdots, n$$

For odd order anti-centrosymmetric matrices,the center of its elements must be 0. Then the sum of the elements on the diagonal is

$$a_{11} + a_{22} + \cdots + a_{nn} = 0$$

Because of all of the above,the trace of a n-order anti-centrosymmetric matrix is 0, and the sum of the eigenvalues is 0, too.

Theorem 4. If A is an n-order square matrix, then there exist the only $B \in CSR^{n\times n}$ and $C \in ACSR^{n\times n}$, which can indicate $A = B + C$.

Proof: Let

$$B = \frac{A + \hat{A}}{2}, C = \frac{A - \hat{A}}{2},$$

then $B \in CSR^{n\times n}$ and $C \in ACSR^{n\times n}$, the uniqueness will be proved in the following.

Suppose

$$A = B_1 + C_1 = B_2 + C_2$$

among which $B_1, B_2 \in CSR^{n\times n}$ and $C_1, C_2 \in ACSR^{n\times n}$,

then

$$B_1 - B_2 = C_1 - C_2$$

However, $B_1 - B_2 \in CSR^{n\times n}$ and $C_1 - C_2 \in ACSR^{n\times n}$, on the basis of Lemma 3,

$$B_1 - B_2 = C_1 - C_2 = O$$

so

$$B_1 = B_2, C_1 = C_2$$

Theorem 5. The product of a anti-centrosymmetric matrix and a same order centrosymmetric matrix is a anti-centrosymmetric matrix; the product of two anti-centrosymmetric matrices of the same order is a centrosymmetric matrix.

Proof: (i)If $A \in ACSR^{n\times n}$ and $B \in CSR^{n\times n}$,then

$$A = -J_n A J_n \text{ and } B = J_n B J_n$$

hence

$$AB = -J_n A J_n J_n B J_n = -J_n (AB) J_n$$

which means that $AB \in ACSR^{n\times n}$.

(ii) If $A, B \in ACSR^{n\times n}$,then

$$A = -J_n A J_n \text{ and } B = -J_n B J_n$$

hence

$$AB = J_n A J_n J_n B J_n = J_n (AB) J_n$$

which means that $AB \in CSR^{n\times n}$.

Theorem 6. Odd order anti-centrosymmetric matrices are singular.

Proof: Suppose $A \in ACSR^{(2n+1)\times(2n+1)}$ and $\lambda_1, \lambda_2, \cdots, \lambda_n, \lambda_{n+1}$ is the $n+1$ eigenvalues of it.According to Theorem 3,

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n + \lambda_{n+1} = 0$$

From Theorem 1, there are n pairs eigenvalues which is opposite with each other in $\lambda_1, \lambda_2, \cdots, \lambda_n, \lambda_{n+1}$, and there must be a eigenvalue $\lambda_i = 0$ left. Then 0 must be a eigenvalue of A, which implicates that $\det A = 0$. Consequently, odd order anti-centrosymmetric matrices are singular.

In this paper, the nonsingular anti-centrosymmetric matrices can only be considered in the field of even order matrices.

Theorem 7. The converse of a nonsingular anti-centrosymmetric matrix is anti-centrosymmetric.

Proof: From what is given, $A = -J_n A J_n$ and $AA^{-1} = I_n$, then

$$J_n A A^{-1} J_n = J_n A J_n J_n A^{-1} J_n = A(-J_n A^{-1} J_n) = I_n$$

Above with the uniqueness of A^{-1}, $A^{-1} = -J_n A^{-1} J_n$ can be deduced, and that is A^{-1} is also anti-centrosymmetric.

Theorem 8. If the nonsingular matrix $A \in ACSR^{n\times n}$ and $B=(A \vdots E_k)$, among which $n=2k, k\in Z^*$ and E_k is the first k columns of n-order identity matrix. Using elementary transformation to B, if A becomes identity matrix and meanwhile E_k becomes matrix C, then $A^{-1}=(C,-\hat{C})$.

Proof: In view of Theorem 7, if $A \in ACSR^{n\times n}$ and A is nonsingular, then $A^{-1}\in ACSR^{n\times n}$. Let $A^{-1}=(A_1, A_2)$, among which A_1 is the matrix which is constituted by the first k columns of A^{-1}, and A_2 is the matrix which is constituted by the end k columns of A^{-1}. The same block-divided method can be used to I_n, and let $A(A_1, A_2)=(E_k, G)$.

Now,

$$(AA_1, AA_2)=(E_k, G)$$

so

$$AA_1 = E_k$$

If A becomes I_n by elementary transformation, then there exist elementary matrices $P_1, P_2, \cdots, P_m$, which can infer

$$P_1P_2\cdots P_mA=I_n.$$

Consequently,

$$A^{-1}=P_1P_2\cdots P_m \text{ and } A_1=A^{-1}E_k=P_1P_2\cdots P_mE_k.$$

From all of the above, if elementary transformation is used to $B=(A\vdots E_k)$, and A

becomes the identity matrix, then E_k becomes the first k columns of A^{-1}. Finally, according to Lemma 4(2), the conclusion can be obtained.

Theorem 9. If $A\in ACSR^{2n\times 2n}$ and that is $A=\begin{pmatrix} A_1 & A_2J_n \\ -J_nA_2 & -J_nA_1J_n \end{pmatrix}$, then A is nonsingular if and only if $B=A_1+A_2$ and $C=A_1-A_2$ are all nonsingular, besides,

$$A^{-1}=\frac{1}{2}\begin{pmatrix} B^{-1}+C^{-1} & (C^{-1}-B^{-1})J_n \\ -J_n(C^{-1}-B^{-1}) & -J_n(B^{-1}+C^{-1})J_n \end{pmatrix}$$

among which A_1, A_2 are n-order square matrices, and J_n is n-order permutation matrix.

Proof: Let $U=\frac{1}{\sqrt{2}}\begin{pmatrix} I_n & -I_n \\ J_n & J_n \end{pmatrix}$, then

$$U^TAU=\begin{pmatrix} O & -C \\ -B & O \end{pmatrix} \square \begin{pmatrix} A_1 & A_2J_n \\ -J_nA_2 & -J_nA_1VJ_n \end{pmatrix}=A$$

and $|A|=-|B||C|$. Therefore, A is nonsingular if and only if B and C are nonsingular.

Moreover,

$$\begin{pmatrix} A_1 & A_2 J_n \\ -J_n A_2 & -J_n A_1 J_n \end{pmatrix} \frac{1}{2} \begin{pmatrix} B^{-1}+C^{-1} & (C^{-1}-B^{-1})J_n \\ -J_n(C^{-1}-B^{-1}) & -J_n(B^{-1}+C^{-1})J_n \end{pmatrix} = \begin{pmatrix} I_n & O \\ O & I_n \end{pmatrix}$$

so

$$A^{-1} = \frac{1}{2} \begin{pmatrix} B^{-1}+C^{-1} & (C^{-1}-B^{-1})J_n \\ -J_n(C^{-1}-B^{-1}) & -J_n(B^{-1}+C^{-1})J_n \end{pmatrix}$$

Example: If $A = \begin{pmatrix} 3 & 7 & -3 & -1 \\ 2 & 3 & 1 & 1 \\ -1 & -1 & -3 & -2 \\ 1 & 3 & -7 & -3 \end{pmatrix}$, then $A^{-1} = ?$

Solve: 1. According to Theorem 8,

$$(A \vdots E) = \begin{pmatrix} 3 & 7 & -3 & -1 & \vdots & 1 & 0 \\ 2 & 3 & 1 & 1 & \vdots & 0 & 1 \\ -1 & -1 & -3 & -2 & \vdots & 0 & 0 \\ 1 & 3 & -7 & -3 & \vdots & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -7 & -3 & \vdots & 0 & 0 \\ -1 & -1 & -3 & -2 & \vdots & 0 & 0 \\ 2 & 3 & 1 & 1 & \vdots & 0 & 1 \\ 3 & 7 & -3 & -1 & \vdots & 1 & 0 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 3 & -7 & -3 & \vdots & 0 & 0 \\ 0 & 1 & -5 & -5/2 & \vdots & 0 & 0 \\ 0 & -3 & 15 & 7 & \vdots & 0 & 1 \\ 0 & -2 & 18 & 8 & \vdots & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & \vdots & -1 & 3 \\ 0 & 1 & 0 & 0 & \vdots & 5/8 & -5/4 \\ 0 & 0 & 1 & 0 & \vdots & 1/8 & 3/4 \\ 0 & 0 & 0 & 1 & \vdots & 0 & -2 \end{pmatrix} = (I, C)$$

then $A^{-1} = (C, -\hat{C}) = \begin{pmatrix} -1 & 3 & 2 & 0 \\ 5/8 & -5/4 & -3/4 & -1/8 \\ 1/8 & 3/4 & 5/4 & -5/8 \\ 0 & -2 & -3 & 1 \end{pmatrix}$

2. According to Theorem 9,

$$A_1 = \begin{pmatrix} 3 & 7 \\ 2 & 3 \end{pmatrix}, A_2 J = \begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix}$$

then

$$A_1 + A_2 = \begin{pmatrix} 2 & 4 \\ 3 & 4 \end{pmatrix} = B, A_1 - A_2 = \begin{pmatrix} 4 & 10 \\ 1 & 2 \end{pmatrix} = C$$

and

$$B^{-1} = \begin{pmatrix} -1 & 1 \\ 3/4 & -1/2 \end{pmatrix}, C^{-1} = \begin{pmatrix} -1 & 5 \\ 1/2 & -2 \end{pmatrix}$$

so

$$A^{-1}=\frac{1}{2}\begin{pmatrix} B^{-1}+C^{-1} & (C^{-1}-B^{-1})J \\ -J(C^{-1}-B^{-1}) & -J(B^{-1}+C^{-1})J \end{pmatrix}=\begin{pmatrix} -1 & 3 & 2 & 0 \\ 5/8 & -5/4 & -3/4 & -1/8 \\ 1/8 & 3/4 & 5/4 & -5/8 \\ 0 & -2 & -3 & 1 \end{pmatrix}$$

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