Global asymptotic stability in a rational dynamic equation on discrete time scales

$$x(\sigma(t))=\frac{ax(t)-bx(\rho(t))}{c+(x(\rho(t)))^{m}},\quad t\in \mathrm{T}$$

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Abstract- *In this paper, we study the global stability, periodicity character and some other properties of solutions of the rational dynamic equation on discrete time scales*

$$x(\sigma(t))=\frac{ax(t)-bx(\rho(t))}{c+(x(\rho(t)))^{m}},\quad t\in \mathrm{T},$$

where $a, b, c \succ 0$ and $m \geq 1$.

Keywords-Rational dynamic equation, Time scales, Equilibrium point, Global attractor, Periodicity, Boundedness, Invariant interval.

I. INTRODUCTION

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his PhD thesis [15] in order to unify continuous and discrete analysis. The theory of dynamic equations not only unifies the theories of differential equations and difference equations, but also it extends these classical cases to cases in between, e.g., to so-called q-difference equations. Since then several authors have expounded on various aspects of this new theory, see the survey paper by Agarwal et al. [2] and the references cited therein.

Many other interesting time scales exist, and they give rise to many applications, among them the study of population dynamic models (see [8]). A book on the subject of time scales by Bohner and Petreson [5] summarizes and organizes much of the time scales calculus (see also [4]).

The study of rational dynamic equations on time scales goes to back to Elzeiny [13].

For the notions used below we refer the reader to [5] and to the following a short introduction to the time scale calculus.

Definition 1.1: A time scale is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Thus $\mathbb{R},\mathbb{Z},\mathbb{N},\mathbb{N}_0,$ i. e., the real numbers, the integers, the natural numbers, and the nonnegative integers are examples of time scales, as are $[0,1]\cup[2,3],\ [0,1]\cup\mathbb{N},$ *and the Cantor set, while* $\mathbb{Q},\ \mathbb{R}\setminus\mathbb{Q},\ \mathbb{C},(0,1),$ i. e., the rational numbers, the irrational numbers, the complex numbers, and the open interval between 0 and 1, are not time scales. Throughout this paper, a time scale is denoted by the symbol T and has the topology that it inherits from the real numbers with the standard topology.

To reference points in the set T, the forward and backward jump operators are defined.

Definition 1.2: For $t\in T,$, the forward operator $\sigma: T\to T$ is defined by

$$\sigma(t)=\inf\{s\in T: s\succ t\},$$

and backward operator $\rho: T\to T$ is defined by

$$\rho(t)=\sup\{s\in T: s\prec t\}.$$

If T has a maximum t^* and a minimum $t^\bullet$, then

$\sigma(t^*) = t^*,\ and\ \rho(t^\bullet) = t^\bullet$. When $\sigma(t) \neq t$, then t is called right scattered. When $\rho(t) \neq t$, then t is called left scattered.

Points t such that

$$\rho(t) \prec t \prec \sigma(t),\ \rho(t) \prec t = \sup T,\ or\ \sigma(t) \succ t = \inf T,$$

are called isolated points. If a time scale consists of only isolated points, then it is an isolated (discrete) time scale. Also, if $t \prec \sup T\ and\ \sigma(t) = t,\ then\, t$ is called right-dense, and if $t \succ \inf T\ and\ \rho(t) = t,\ then\ t$ is called left-dense. Points t that are either left-dense or right-dense are called dense.

Finally, the graininess operator $\mu : T \to [0, \infty)\ is\ defined\ by\ \mu(t) = \sigma(t) - t$

and if $f : T \to \mathbb{R}$ is a function, then the function $f^\sigma : T \to \mathbb{R}$ is defined by

$$f^\sigma(t) = f(\sigma(t))\ \ for\ all\ t \in T,$$

$i.e.,\ \ f^\sigma = f \circ \sigma$ is the composition function of f with σ.

Example 1.1: Let us briefly consider the two examples $T = \mathbb{R}\ and\ T = \mathbb{Z}$.

(i) If $T = \mathbb{R}$, then we have for any $t \in \mathbb{R}$, $\sigma(t) = \inf\{s \in \mathbb{R} : s \succ t\} = \inf(t, \infty) = t,$

and similarly $\rho(t) = t.$ Hence every point $t \in \mathbb{R}$ is left-dense and right-dense.

The graininess operator $\mu = 0\ \ for\, all\ \ t \in \mathbb{R}.$

(ii) If $T = \mathbb{Z}$, then we have for any $t \in \mathbb{Z}$

$\sigma(t) = \inf\{s \in \mathbb{Z} : s \succ t\} = \inf\{t+1, t+2, ...\} = t+1,$ and similarly $\rho(t) = t - 1.$ Hence, every point $t \in \mathbb{Z}$ is left-scattered and right-scattered. The graininess operator $\mu = 1\ \ for\ all\ \ t \in \mathbb{Z}.$

Example 1.2: Consider the time scale $T = \mathbb{N}_0^2 = \{n^2 : n \in \mathbb{N}_0\}$. Then, we have

$$\sigma(n^2) = (n+1)^2,\ \rho(n^2) = (n-1)^2,\ and\ \ \mu(n^2) = 2n+1\ \ for\ \ n \in \mathbb{N}_0.\ Thus,$$
$$\sigma(t) = (\sqrt{t}+1)^2,\ \rho(t) = (\sqrt{t}-1)^2,\ and\ \ \mu(t) = 2\sqrt{t}+1.$$

Example 1.3: Let $T = \mathbb{Z}/4 = \{k/4 : k \in \mathbb{Z}\}$. Then, we have for $t \in T$,

$$\sigma(t) = \inf\{s \in T : s \succ t\} = \inf\{t + n/4 : n \in \mathbb{N}\} = t + 1/4,\ \ and\ \ \rho(t) = t - h/4.$$

Hence, every point $t \in T$ is isolated and $\mu(t) = \sigma(t) - t = t + 1/4 - t = 1/4\ \ for\ \ t \in T,$

so that μ in this example is constant.

Example 1.4: Consider the time scale $T = 2^{\mathbb{N}},\ \ we\ get\ \sigma(t) = 2t,\ \ \rho(t) = \frac{t}{2},\ and\ \ \mu(t) = t\ \ for\ \ t \in T.$

Continuity on time scales is defined in the following manner.

Definition 1.3: Assume $f : T \to \mathbb{R}$ is a function and let $t \in T$. If t is an isolated point, then we define

$$\lim_{s \to t} f(s) = f(t)$$

and we say f is continuous at t. When t is not isolated point, then when we write

$$\lim_{s \to t} f(s) = L,$$

it is understood that s approaches t in the time scale $(s \in T,\ s \neq t)$. We say f is continuous on $T.$ provided

$$\lim_{s \to t} f(s) = f(t) \ \ for\ all\ t \in T.$$

In particular, we have that any function defined on an isolated time scale (since all of its points are isolated points) is continuous. We say $f:[a,b]_T \to \mathbb{R}$ is continuous provided f is continuous at each point in $(a,b)_T$, f is left continuous at a, and f is right continuous at b.

In the time scale calculus, the functions are right-dense continuous which we now define.

Definition 1.4: A function $f:T \to \mathbb{R}$ is called right-dense continuous or briefly rd-continuous provided it is continuous at right-dense points in $T.$ and its left-sided limits exist (finite) at left-dense points in $T.$. The set of rd-continuous functions $f:T \to \mathbb{R}$ will be denoted in this paper by $C_{rd} = C_{rd}(T) = C_{rd}(T,\mathbb{R})$

In the next theorem we see that the jump operator is rd-continuous.

Theorem 1.1 (Bohner et al. [5]): The forward operator $\sigma : T \to T$ is increasing, rd-continuous, and

$$\sigma(t) \geq t \ \ for\ all\ t \in T,$$

and the jump operator is discontinuous at points which are left-dense and right-scattered.

Remark 1.1: The graininess function $\mu : T \to [0,\infty)$ is rd-continuous and μ is discontinuous at points in T that are both left-dense and right-scattered.

When $T = \mathbb{Z}$, the rational dynamic equation on discrete time scales

$$x(\sigma(t)) = \frac{ax(t) - bx(\rho(t))}{c + (x(\rho(t)))^m}, \ \ t \in \mathrm{T},$$

$where\ a,b,c \succ 0\ and\ m \geq 1.$ becomes the recursive sequence

$$x_{n+1} = \frac{ax_n - bx_{n-1}}{c + x_{n-1}^m},\ n = 1,2,..., \tag{1.1}$$

$where\ a,b,c > 0\ and\ m \geq 1.$.

Now, the difference equations (as well as differential equations and delay differential equations) model various diverse phenomena in biology, ecology, physiology, physics, engineering and economics, etc.[20]. The study of nonlinear difference equations is of paramount importance not only in their own right but in understanding the behavior of their differential counterparts.

There is a class of nonlinear difference equations, known as the rational difference equations, each of which consists of the ratio of two polynomials in the sequence terms. There has been a lot of work concerning the global asymptotic behavior of solutions of rational difference equations [1, 3, 6, 7, 9- 12, 14, 16, 18, 19, 21-26].

This paper addresses, the global stability, periodicity character and boundedness of the solutions of the rational dynamic equation on discrete time scales

$$x(\sigma(t)) = \frac{ax(t) - bx(\rho(t))}{c + (x(\rho(t)))^m},\ t \in T, \tag{1.2}$$

$where\ a, b, c \succ 0\ and\ m \geq 1.$

When $T = \mathbb{Z}\ and\ m = 1$, our equation reduces to equation which examined by Yang et al. [24].

Here, we recall some notations and results which will be useful in our investigation.

Let I be some interval of real numbers and let f be a continuous function defined on $I \times I$. Then, for initial conditions $x(\rho(t_0)), x(t_0) \in I$, it is easy to see that the dynamic equation on discrete time scales

$$x(\sigma(t)) = f(x(t), x(\rho(t))),\ t \in T, \tag{1.3}$$

has a unique solution $\{x(t) : t \in T\}$, which is called a recursive sequence on time scales.

Definition 1.5: A point ω is called an equilibrium point of equation (1.3) if

$\omega = f(\omega, \omega).$

That is, $x(t) = \omega\ for\ t \in T$, is a solution of equation (1.3), or equivalently, ω is a fixed point of f.

Assume ω is an equilibrium point of equation (1.3) and $u = -f_{x(t)}(\omega, \omega),\ and\ v = -f_{x(\rho(t))}(\omega, \omega)$. Then the linearized equation associated with equation (1.3) about the equilibrium point ω is

$$z_{\sigma(t)} + uz_t + vz_{\rho(t)} = 0. \tag{1.4}$$

The characteristic equation associated with equation (1.4) is

$$\lambda^2 + u\lambda + v = 0. \tag{1.5}$$

Theorem 1.2 (Linearized stability theorem [17]):

(1) $If\ |u| \prec 1 + v\ and\ v \prec 1,\ then\ \omega\ is\ locally\ asymptotically\ stable.$

(2) $If\ |u| \prec |1 + v|\ and\ |v| \prec 1,\ then\ \omega\ is\ a\ repeller.$

(3) $If\ |u| \succ |1 + v|\ and\ u^2 \succ 4v,\ then\ \omega\ is\ a\ saddle\ point.$

(4) $If\ |u| = |1 + v|,\ then\ \omega\ is\ a\ non-hyperbolic\ point.$

Definition 1.6: We say that a solution $\{x(t) : t \in T\}$ of equation (1.3) is bounded if

$|x(t)| \prec A\ \ for\ all\ t \in T.$

Definition 1.7: (a) A solution $\{x(t) : t \in T\}$ of equation (1.2) is said to be periodic with period υ if

$$x(t + \upsilon) = x(t)\ \ for\ all\ t \in T. \tag{1.6}$$

(b) A solution $\{x(t) : t \in T\}$ of equation (1.3) is said to be periodic with prime period υ, or υ-cycle if it is periodic with period υ and υ is the least positive integer for which (1.6) holds.

Definition 1.8: An interval $J \subseteq I$ is called invariant for equation (1.3) if every solution $\{x(t) : t \in T\}$ of equation (1.3) with initial conditions $(x(\rho(t_0)), x(t_0)) \in J \times J$ satisfies $x(t) \in J\ \ for\ all\ t \in T.$

For a real number $x(t_0)$ and a positive number R, let $O(x(t_0), R) = \{x(t) :| x(t) - x(t_0)| \prec R\}.$

For other basic terminologies and results of difference equations the reader is referred to [17].

II. MAIN RESULTS

2.1 Local asymptotic stability of the equilibrium points

Consider the rational dynamic equation on discrete time scales

$$x(\sigma(t)) = \frac{ax(t) - bx(\rho(t))}{c + (x(\rho(t)))^m},\ t \in T, \tag{2.1}$$

where $a, b, c > 0$ *and* $m \geq 1$..

Let $a' = \frac{a}{b},\ c' = \frac{c}{b},\ and\ d' = \frac{1}{b}.$ Then equation (2.1) can be rewritten as

$$x(\sigma(t)) = \frac{a'x(t) - x(\rho(t))}{c' + d'(x(\rho(t)))^m},\ t \in T. \tag{2.2}$$

The change of variables $y(t) = (d')^{1/m} x(t),$ followed by the change $x(t) = y(t)$ reduces the above equation to

$$x(\sigma(t)) = \frac{px(t) - x(\rho(t))}{q + (x(\rho(t)))^m},\ t \in T, \tag{2.3}$$

where $p = a' = \frac{a}{b},\ and\ q = c' = \frac{c}{b}.$ Hereafter, we focus our attention on equation (2.3) instead of equation (2.1).

Now, when m is the ratio of odd positive integers, equation (2.3) has only two equilibrium points

$$\alpha = 0,\ and\ \beta = \sqrt[m]{p - q - 1},$$

and when m is positive rational number and numerator's even positive integers, equation (2.3) has unique equilibrium point

$$\alpha = 0\ if\ \ p \leq q + 1,$$

and three equilibrium points:

$\alpha = 0, \beta = \sqrt[m]{p - q - 1}, and\ \gamma = -\sqrt[m]{p - q - 1}\ if\ \ p \succ q + 1.$ Furthermore, suppose that, $m = \frac{2k+1}{2n},\ \ k, n \in \mathbb{Z}_+.$

Then, in this case, we consider $x(t) \geq 0\ \ for\ all\ \ t \in T.$

When $p \leq q + 1,$ equation (2.3) has unique equilibrium point

$$\alpha = 0.$$

When $p \succ q + 1,$ however, equation (2.3) has the following two equilibrium points:

$$\alpha = 0,\ and\ \beta = \sqrt[m]{p - q - 1}.$$

The local asymptotic behavior of $\alpha = 0$ is characterized by the following result.

THEOREM 2.1:

(1) If $q \succ \max\{1, p - 1\}$, then α is locally asymptotically stable.

(2) If $p - 1 \prec q \prec 1,$ then α is a repeller.

(3) If $q \prec p - 1$, then α is a saddle point.

Proof: The Linearized equation associated with equation (2.3) about the equilibrium point $\alpha = 0$ is

$$z(\sigma(t)) - \frac{p}{q} z(t) + \frac{1}{q} z(\rho(t)) = 0, \ t \in T. \tag{2.4}$$

Now, when $T = \mathbb{Z}$, then equation (2.4) becomes

$$z_{n+1} - \frac{p}{q} z_n + \frac{1}{q} z_{n-1} = 0, \ n \in \mathbb{Z}. \tag{2.5}$$

The characteristic equation associated with equation (2.5) is

$$\lambda^2 - \frac{p}{q}\lambda + \frac{1}{q} = 0, \ where \ z(t) = \lambda^t, \tag{2.6}$$

when $T = r^{\mathbb{N}}, \ r \succ 1,$ then equation (2.4) becomes

$$z_{r^{n+1}} - \frac{p}{q} z_{r^n} + \frac{1}{q} z_{r^{n-1}} = 0, \ n \in \mathbb{N}. \tag{2.7}$$

The characteristic equation associated with equation (2.7) is

$$\lambda^2 - \frac{p}{q}\lambda + \frac{1}{q} = 0, \ where \ z(t) = \lambda^{\log_r(t)}, \tag{2.8}$$

when $T = h\mathbb{Z},$ then equation (2.4) becomes

$$z_{h(n+1)} - \frac{p}{q} z_{hn} + \frac{1}{q} z_{h(n-1)} = 0, \ n \in \mathbb{Z}. \tag{2.9}$$

The characteristic equation associated with equation (2.9) is

$$\lambda^2 - \frac{p}{q}\lambda + \frac{1}{q} = 0, \ where \ z(t) = \lambda^{t/h}, \tag{2.10}$$

and when $T = \mathbb{N}_0^2,$ then equation (2.4) becomes

$$z_{(n+1)^2} - \frac{p}{q} z_{n^2} + \frac{1}{q} z_{(n-1)^2} = 0, \ n \in \mathbb{N}_0. \tag{2.11}$$

The characteristic equation associated with equation (2.9) is

$$\lambda^2 - \frac{p}{q}\lambda + \frac{1}{q} = 0, \ where \ z(t) = \lambda^{\sqrt{t}}. \tag{2.12}$$

Hence, the characteristic equation associated with equation (2.4) is

$$\lambda^2 - \frac{p}{q}\lambda + \frac{1}{q} = 0, \ for \ all \ t \in T. \tag{2.13}$$

$$Let \ u = -\frac{p}{q}, \ and \ v = \frac{1}{q}.$$

(1) The result follows from Theorem 1.2 (1) and the following relations

$$|u|-(1+v)=\frac{p}{q}-(1+\frac{1}{q})=\frac{(p-1)-q}{q}\prec 0,\ and\ \ v=\frac{1}{q}\prec 1.$$

(2) The result follows from Theorem 1.2(2) and the following relations

$$|u|-|1+v|=\frac{p}{q}-|1+\frac{1}{q}|=\frac{p}{q}-|\frac{q+1}{q}|=\frac{(p-1)-q}{q}\prec 0,\ and\ |v|=\frac{1}{q}\succ 1.$$

(3) Note that $q\prec p-1\le p^2/4.$ The result follows from Theorem 1.2 (3) and the following relations

$$|u|-|1+v|=\frac{p}{q}-|1+\frac{1}{q}|=\frac{p}{q}-|\frac{q+1}{q}|=\frac{(p-1)-q}{q}\succ 0,\ and\ u^2-4v=(\frac{p}{q})^2-\frac{4}{q}=\frac{p^2-4q}{q^2}\succ 0.$$

Now, the local asymptotic behavior of β and γ are characterized by the following result.

Theorem 2.2 : Assume that $p\neq 1.$

(i) If either $m(p-q-1)\prec p-2\ or\ 3p-1\prec q\prec 1, where\ m=1,$

then both $\beta\ and\ \gamma$ are locally asymptotically stable.

(ii) If either $m(p-q-1)\succ p-2\, or\, q\succ \max\{1,\ 3p-1\}, where\ m=1,$

then both $\beta\ and\ \gamma$ are repeller.

(iii) If either $p-1\prec q\prec 3p-1, where\ m=1,$ then β is a saddle point.

Proof : The Linearized equation associated with equation (2.3) about the equilibrium β is

$$z(\sigma(t))-\frac{p}{p-1}z(t)+\frac{m(p-q-1)+1}{p-1}z(\rho(t))=0. \qquad (2.14)$$

Hence, the characteristic equation associated with equation (2.14) is

$$\lambda^2-\frac{p}{p-1}\lambda+\frac{m(p-q-1)+1}{p-1}=0, \forall t\in T.$$

$$Let\ u=\frac{-p}{p-1}\ \ and\ \ v=\frac{m(p-q-1)+1}{p-1}.$$

(i) Assume $m(p-q-1)\prec p-2,\ then,\ p\succ 1.$ Then, from Theorem 1.2 (1) and the following relations

$$|u|-(v+1)=|\frac{p}{p-1}|-(\frac{m(p-q-1)+1}{p-1}+1)=\frac{p}{p-1}-(\frac{m(p-q-1)+p}{p-1})=\frac{-m(p-q-1)}{p-1}\prec 0,$$

and

$$v=\frac{m(p-q-1)+1}{p-1}\prec 1,$$

we conclude that β is locally asymptotically stable. Assume $3p-1\prec q\prec 1,\ where\ m=1,\ then,\ p\prec 1.$ Then, from Theorem 1.2 (1) and the following relations

$$|u|-(v+1)=|\frac{p}{p-1}|-(\frac{(p-q)}{p-1}+1)=\frac{p}{1-p}-(\frac{(p-q)+p-1}{p-1})=\frac{3p-1-q}{1-p}\prec 0,$$

and

$$v=\frac{p-q}{p-1}=\frac{q-p}{1-p}\prec 1,$$

we conclude that β is locally asymptotically stable. Similarly, we can prove that γ is locally asymptotically stable.

(ii) Assume $m(p-q-1)\succ p-2, \textit{ then, } p\succ 1.$ Then, from Theorem 1.2 (2) and the following relations

$$|u|-(v+1)=|\frac{p}{p-1}|-(\frac{m(p-q-1)+1}{p-1}+1)=\frac{p}{p-1}-(\frac{m(p-q-1)+p}{p-1})=\frac{-m(p-q-1)}{p-1}\prec 0,$$

and

$$|v|=\frac{m(p-q-1)+1}{p-1}\succ 1,$$

we conclude that β is a repeller. Assume that $q\succ \max\{1,3p-1\}, \textit{where } m=1.$ Then, from Theorem 1.2 (2) and the following relations

$$|u|-(v+1)=|\frac{p}{p-1}|-|\frac{(p-q)}{p-1}+1|=\frac{p}{|p-1|}-|\frac{2p-q-1}{p-1}|=\frac{(3p-1)-q}{|p-1|}\prec 0,$$

and when $p\succ 1, \textit{ we have } q-p\succ 2p-1>0, \textit{so}$

$$|v|=|\frac{p-q}{p-1}|=\frac{q-p}{p-1}\succ 1,$$

when $p\prec 1, \textit{ we have } q\succ 1\succ p, \textit{so}$

$$|v|=|\frac{p-q}{p-1}|=\frac{q-p}{1-p}\succ 1,$$

we conclude that β is a repeller. Similarly, we can prove that γ is a repeller.

(iii) If $p\succ 1.$ Note that the given condition is equivalent to $p\succ|2p-q-1|.$ Besides, we have $q\succ p-1\geq\frac{p(3p-4)}{4(p-1)}.$ Then, from Theorem 1.2 (3) and the following relations

$$|u|-|v+1|=|\frac{p}{p-1}|-|\frac{p-q}{p-1}+1|=|\frac{p}{p-1}|-\frac{|2p-q-1|}{p-1}\succ 0, \textit{ and}$$

$$u^2-4v=(\frac{p}{p-1})^2-\frac{4(p-q)}{p-1}=\frac{1}{(p-1)^2}[4(p-1)q-p(3p-4)]\succ 0,$$

we conclude that β is a saddle point.

Assume that $p \prec 1.$ Note that the given condition is equivalent to $p \succ |2p-q-1|.$ Besides, we have $q \prec 3p-1 \le \frac{p(4-3p)}{4(1-p)}.$ Then, from Theorem 1.2 (3) and the following relations

$$|u|-|v+1|=|\frac{p}{p-1}|-|\frac{p-q}{p-1}+1|=\frac{p}{1-p}-\frac{|2p-q-1|}{1-p}\succ 0, \text{ and}$$

$$u^2-4v=(\frac{p}{p-1})^2-\frac{4(p-q)}{p-1}=\frac{1}{(1-p)^2}[p(4-3p)-4(1-p)q]\succ 0,$$

we conclude that β is a saddle point.

2.2 Boundedness of solutions of equation (2.3)

In this section, we study the boundedness of solution of equation (2.3).

Theorem 2.3: Suppose that $p+1 \le q$ and m is positive rational number and numerator's even positive integers, $i=1,2$. Then the solution of equation (2.3) is bounded for all $t \in T$.

Proof: We argue that $|x(t)| \prec A$ for all $t \in T$ by induction on t.

Case (1): If $T=\mathbb{Z}$. Given any initial conditions $|x_{-1}| \prec A$ and $|x_0| \prec A$, we argue that $|x_n| \prec A$ for all $n \in \mathbb{Z}$ by induction on n. It follows from the given initial conditions that this assertion is true for $n=-1,0.$ Suppose the assertion is true for $n-2$ and $n-1 (n \ge 1).$ That is,

$$|x_{n-2}| \prec A \text{ and } |x_{n-1}| \prec A.$$

Now, we consider x_n, where we put n instead of $(n+1)$ in equation (2.3),

$$|x_n| \le \frac{|px_{n-1}-x_{n-2}|}{|q+x_{n-2}^m|} \le \frac{p|x_{n-1}|+|x_{n-2}|}{|q+x_{n-2}^m|} \le \frac{(p+1)A}{|q+x_{n-2}^m|} \tag{2.15}$$

Since,

$$|q+x_{n-2}^m| = q+x_{n-2}^m \ge q \succ 0, \frac{1}{|q+x_{n-2}^m|} \le \frac{1}{q}. \tag{2.16}$$

From (2.15) and (2.16), we obtain,

$$|x_n| \prec \frac{(p+1)A}{q} \prec A \text{ for all } n.$$

Case(2): If $T=h\mathbb{Z}=\{hk : h \succ 0 \text{ and } k \in \mathbb{Z}\}.$ Given any initial conditions $|x_{-h}| \prec A$ and $|x_0| \prec A$, we argue that $|x_{hn}| \prec A$ for all $n \in \mathbb{Z}$ by induction on n. It follows from the given initial conditions that this assertion is true for $n=-1,0.$ Suppose the assertion is true for $n-2$ and $n-1 (n \ge 1).$ That is,

$$|x_{h(n-2)}| \prec A \text{ and } |x_{h(n-1)}| \prec A.$$

Now, we consider x_{hn}, where we put hn instead of $h(n+1)$ in equation (2.3),

$$|x_{hn}| \le \frac{|px_{h(n-1)} - x_{h(n-2)}|}{|q + x^m_{h(n-2)}|} \le \frac{p|x_{h(n-1)}| + |x_{h(n-2)}|}{|q + x^m_{h(n-2)}|} \prec \frac{(p+1)A}{|q + x^m_{h(n-2)}|}. \quad (2.17)$$

Since, $$|q + x^m_{h(n-2)}| = q + x^m_{h(n-2)} \ge q \succ 0, \ \frac{1}{|q + x^m_{h(n-2)}|} \le \frac{1}{q}. \quad (2.18)$$

From (2.17) and (2.18), we obtain,

$$|x_{hn}| \prec \frac{(p+1)A}{q} \prec A \ \textit{for all n.}$$

Case (3): If $T = \mathbb{N}_0^2 = \{n^2 : n \in \mathbb{N}_0\}$. Given any initial conditions $|x_1| \prec A \ and \ |x_0| \prec A$, we argue that $|x_k| \prec A \ for\,all \ k = n^2, \ n \in \mathbb{N}_0$ by induction on k. It follows from the given initial conditions that this assertion is true for $k = 0,1.$ Suppose the assertion is true for $(n-2)^2 \ and \ (n-1)^2.$ That is,

$$|x_{(n-2)^2}| \prec A \ and \ |x_{(n-1)^2}| \prec A.$$

Now, we consider x_{n^2}, where we put n instead of $(n+1)$ in equation (2.3),

$$|x_{n^2}| \le \frac{|px_{(n-1)^2} - x_{(n-2)^2}|}{|q + x^m_{(n-2)^2}|} \le \frac{p|x_{(n-1)^2}| + |x_{(n-2)^2}|}{|q + x^m_{(n-2)^2}|} \prec \frac{(p+1)A}{|q + x^m_{(n-2)^2}|}. \quad (2.19)$$

Since, $$|q + x^m_{(n-2)^2}| = q + x^m_{(n-2)^2} \ge q \succ 0, \ \frac{1}{|q + x^m_{(n-2)^2}|} \le \frac{1}{q}. \quad (2.20)$$

From (2.19) and (2.20), we obtain,

$$|x_k| \prec \frac{(p+1)A}{q} \prec A \ \textit{for all k.}$$

Case(4): If $T = r^{\mathbb{N}}, r \succ 1.$ Given any initial conditions $|x_{1/r}| \prec A \ and \ |x_1| \prec A$, we argue that $|x_k| \prec A \ for\,all \ k \in r^n, \ n \in \mathbb{N}$ by induction on k. It follows from the given initial conditions that this assertion is true for $k = 1/r, 1.$ Suppose the assertion is true for $k = r^{n-2} \ and \ k = r^{n-1}, \ where \ n \in \mathbb{N}.$ That is,

$$|x_{r^{n-2}}| \prec A \ and \ |x_{r^{n-1}}| \prec A.$$

Now, we consider x_{r^2}, where we put n instead of $(n+1)$ in equation (2.3),

$$|x_{r^n}| \le \frac{p|x_{r^{n-1}}| + |x_{r^{n-2}}|}{|q + x^m_{r^{n-2}}|} \prec \frac{(p+1)A}{|q + x^m_{r^{n-2}}|}. \quad (2.21)$$

Since, $$|q + x^m_{r^{n-2}}| = q + x^m_{r^{n-2}} \ge q, \ \frac{p+1}{|q + x^m_{r^{n-2}}|} \le \frac{1}{q}. \quad (2.22)$$

From (2.21) and (2.22), we obtain,

$$|x_k| \prec \frac{(p+1)A}{q} \prec A \ \textit{for all k.This}$$ completes the proof.

2.3 Periodic solutions of equation

In this section, we study the existence of periodic solutions of equation (2.3).

The following theorem states the necessary and sufficient conditions that this equation has a periodic solutions.

Theorem 2.4: Equation (2.3) has prime period two solutions if and only if

$$q \succ 3p-1, where\ q \neq p^2+p\ and\ m=1.$$

Proof: First, suppose that, there exists a prime period two solution

$$\ldots,\varphi,\psi,\varphi,\psi,\ldots,$$

of equation (2.3). We will prove that $q \succ 3p-1.$ We see from equation (2.3) that

$$\varphi = \frac{p\psi-\varphi}{q+\varphi}\ and\ \psi = \frac{p\varphi-\psi}{q+\psi}.$$

Then, $\varphi q+\varphi^2 = p\psi-\varphi,\ and\ \psi q+\psi^2 = p\varphi-\psi,\ hence,$

$$q(\varphi-\psi)+(\varphi-\psi)(\varphi+\psi)= p(\psi-\varphi)+(\psi-\varphi).$$

Since, $\varphi-\psi \neq 0,\ \ q+(\varphi+\psi)=-1-p.$ Therefore,

$$\varphi+\psi=-(p+q+1), \tag{2.23}$$

and so,

$$\varphi-\psi = \frac{pq(\psi-\varphi)+q(\psi-\varphi)+p(\psi-\varphi)(\varphi+\psi)}{(q+\varphi)(q+\psi)}.\ Since,\ -[q^2+q(\varphi+\psi)+\varphi\psi]= pq+q+p(\varphi+\psi).$$

Hence,

$$\varphi\psi=-q(p+q+1)-(p+q)(\varphi+\psi). \tag{2.24}$$

From (2.23) and (2.24), we obtain

$$\varphi\psi=-q(p+q+1)+(p+q)(p+q+1).\ Then,\ \ \varphi\psi = p(p+q+1). \tag{2.25}$$

It is clear now, from equation (2.23) and (2.25) that $\varphi\ and\ \psi$ are the two distinct roots of the quadratic equation

$$t^2+(p+q+1)t+p(p+q+1)=0,$$

So, $(p+q+1)(q+1-3p) \succ 0.\ Then,\ q \succ 3p-1.$

Second, suppose that $q \succ 3p-1.$ We will show that equation (2.3) has a prime period two solutions. Assume that

$$\varphi = \frac{-(p+q+1)+\sqrt{(p+q+1)(q+1-3p)}}{2},\ and\ \psi = \frac{-(p+q+1)-\sqrt{(p+q+1)(q+1-3p)}}{2}.$$

Therefore $\varphi\ and\ \psi$ are distinct real numbers. Set,

$$x(\rho(t_0))=\varphi\ and\ x(t_0)=\psi,\ t_0 \in T.$$

We wish to show that

$$x(\sigma(t_0))=x(\rho(t_0))=\varphi.$$

It follows from equation (2.3) that

$$x(\sigma(t_0))=\frac{px(t_0)-x(\rho(t_0))}{q+x(\rho(t_0))}$$
$$=\frac{p[-(p+q+1)+\sqrt{(p+q+1)(q+1-3p)}}{(q-p-1)-\sqrt{(p+q+1)(q+1-3p)}}+\frac{[(p+q+1)+\sqrt{(p+q+1)(q+1-3p)}]}{(q-p-1)-\sqrt{(p+q+1)(q+1-3p)}}$$
$$=\frac{(q+p+1)(1-p)+(p+1)\sqrt{(p+q+1)(q+1-3p)}}{(q-p-1)-\sqrt{(p+q+1)(q+1-3p)}}$$
$$=\frac{(q+p+1)(1-p)+(p+1)\sqrt{(p+q+1)(q+1-3p)}}{(q-p-1)^2-(p+q+1)(q+1-3p)}\times[(q-p-1)+\sqrt{(p+q+1)(q+1-3p)}]$$
$$=-\frac{2(p^2+p-q)[(p+q+1)+\sqrt{(p+q+1)(q+1-3p)}]}{4(p^2+p-q)}$$
$$=-\frac{(p+q+1)+\sqrt{(p+q+1)(q+1-3p)}}{2}=x(\rho(t_0))=\psi, q\neq p^2+p.$$

Similarly as before one can easily show that

$$x(\sigma(t))=\varphi,\ where\ x(\rho(t))=\varphi\ and\ x(t)=\psi.$$

Thus equation (2.3) has the prime period two solution

$$...,\varphi,\psi,\varphi,\psi,...,$$

where φ *and* ψ are the distinct roots of the quadratic equation (2.3). The proof is completed.

To examine the global attractivity of the equilibrium points of equation (2.3), we first need to determine the invariant intervals for equation (2.3).

2.4 Invariant intervals and global attractivity of the zero equilibria

In this subsection, we determine the family of invariant intervals centered at $\alpha=0$.

Theorem 2.5: Assume $q\succ p+1$. Then for any positive real number

$$A\leq(q-(p+1))^{1/m},$$

The interval $O(0,A)=(-A,A)$ is invariant for eq. (2.3).

Proof: We consider $T=\mathbb{Z},\ h\mathbb{Z}=\{hk:h\succ 0\ and\ k\in\mathbb{Z}\},\ \mathbb{N}_0^2=\{n^2:n\in\mathbb{N}_0\},\ and\ r^{\mathbb{N}},\ r\succ 1.$

Case (1): If $T=\mathbb{Z}$. Given any initial conditions $|x_{-1}|\prec A\ and\ |x_0|\prec A$, we argue that $|x_n|\prec A\ for\ all\ n\in\mathbb{Z}$ by induction on n. It follows from the given initial conditions that this assertion is true for $n=-1,0.$ Suppose the assertion is true for $n-2\ and\ n-1(n\geq 1).$ That is,

$$|x_{n-2}|\prec A\leq(q-(p+1))^{1/m},\ and\ |x_{n-1}|\prec A\leq(q-(p+1))^{1/m}.$$

Now, we consider x_n, where we put n instead of $(n+1)$ in equation (2.3). Since,

$$q+x_{n-2}^m\succ q-A^m\geq q-(q-(p+1))=p+1\succ 0,$$

$$\frac{p+1}{|q+x_{n-2}^m|} \leq 1. \tag{2.26}$$

Thus,

$$|x_n| \leq \frac{|px_{n-1}-x_{n-2}|}{|q+x_{n-2}^m|} \leq \frac{p|x_{n-1}|+|x_{n-2}|}{|q+x_{n-2}^m|} \leq (\frac{p+1}{|q+x_{n-2}^m|}) \times \max\{|x_{n-1}|,|x_{n-2}|\}$$
$$\prec \max\{|x_{n-1}|,|x_{n-2}|\} \prec A \ \ for\,all \ \ n \in \mathbb{Z}.$$

Case (2): If $T = h\mathbb{Z} = \{hk : h \succ 0 \ and \ k \in \mathbb{Z}\}$. Given any initial conditions $|x_{-h}| \prec A \ and \ |x_0| \prec A,$ we argue that $|x_{hn}| \prec A \ \ for\,all \ \ n \in \mathbb{Z}$ by induction on n. It follows from the given initial conditions that this assertion is true for $n = -h, 0.$ Suppose the assertion is true for $n-2 \ and \ n-1 (n \geq 1).$ That is,

$$|x_{h(n-2)}| \prec A \leq (q-(p+1))^{1/m}, \ and \ |x_{h(n-1)}| \prec A \leq (q-(p+1))^{1/m}.$$

Now, we consider x_{hn}, where we put hn instead of $h(n+1)$ in equation (2.3). Since,

$$q + x_{h(n-2)}^m \succ q - A^m \geq q-(q-(p+1)) = p+1 \succ 0,$$

$$\frac{p+1}{|q+x_{h(n-2)}^m|} \leq 1. \tag{2.27}$$

Thus,

$$|x_{hn}| \leq \frac{|px_{h(n-1)}-x_{h(n-2)}|}{|q+x_{h(n-2)}^m|} \leq \frac{p|x_{h(n-1)}|+|x_{h(n-2)}|}{|q+x_{h(n-2)}^m|} \leq (\frac{p+1}{|q+x_{h(n-2)}^m|}) \max\{|x_{h(n-1)}|,|x_{h(n-2)}|\}$$
$$\prec \max\{|x_{h(n-1)}|,|x_{h(n-2)}|\} \prec A \ \ for\,all \ \ n \in \mathbb{Z}.$$

Case (3): If $T = \mathbb{N}_0^2 = \{n^2 : n \in \mathbb{N}_0\}$. Given any initial conditions $|x_1| \prec A \ and \ |x_0| \prec A,$ we argue that $|x_k| \prec A \ \ for\,all \ \ k = n^2, \, n \in \mathbb{N}_0$ by induction on k. It follows from the given initial conditions that this assertion is true for $k = 0,1.$ Suppose the assertion is true for $(n-2)^2 \ and \ (n-1)^2.$ That is,

$$|x_{(n-2)^2}| \prec A \leq (q-(p+1))^{1/m}, \ and \ |x_{(n-1)^2}| \prec A \leq (q-(p+1))^{1/m}.$$

Now, we consider x_{n^2}, where we put n instead of $(n+1)$ in equation (2.3). Since,

$$q + x_{(n-2)^2}^m \succ q - A^m \geq q-(q-(p+1)) = p+1 \succ 0,$$

$$\frac{p+1}{|q+x_{(n-2)^2}^m|} \leq 1. \tag{2.28}$$

Thus,

$$|x_{n^2}| \leq \frac{|px_{(n-1)^2}-x_{(n-2)^2}|}{|q+x_{(n-2)^2}^m|} \leq \frac{p|x_{(n-1)^2}|+|x_{(n-2)^2}|}{|q+x_{(n-2)^2}^m|} \leq (\frac{p+1}{|q+x_{(n-2)^2}^m|}) \max\{|x_{(n-1)^2}|,|x_{(n-2)^2}|\}$$
$$\prec \max\{|x_{(n-1)^2}|,|x_{(n-2)^2}|\} \prec A \ \ for\,all \ \ k = n^2.$$

Case (4): If $T = r^{\mathbb{N}},\ r \succ 1$. Given any initial conditions $|x_{1/r}| \prec A\ and\ |x_1| \prec A,$ we argue that $|x_k| \prec A\ \ for\,all\ \ k \in r^n,\ n \in \mathbb{N}$ by induction on k. It follows from the given initial conditions that this assertion is true for $k = 1/r, 1.$ Suppose the assertion is true for $k = r^{n-2}\ \ and\ \ k = r^{n-1},\ where\ n \in \mathbb{N}.$ That is,

$$|x_{r^{n-2}}| \prec A \le (q-(p+1))^{1/m},\ and\ |x_{r^{n-1}}| \prec A \le (q-(p+1))^{1/m}.$$

Now, we consider x_{r^2}, where we put n instead of $(n+1)$ in equation (2.3). Since,

$$q + x^m_{r^{n-2}} \succ q - A^m \ge q-(q-(p+1)) = p+1 \succ 0,$$

$$\frac{p+1}{|q+x^m_{r^{n-2}}|} \le 1. \tag{2.29}$$

Thus,

$$|x_{r^n}| \le \frac{|px_{r^{n-1}} - x_{r^{n-2}}|}{|q+x^m_{r^{n-2}}|} \le \frac{p|x_{r^{n-1}}|+|x_{r^{n-2}}|}{|q+x^m_{r^{n-2}}|} \le \left(\frac{p+1}{|q+x^m_{r^{n-2}}|}\right)\max\{|x_{r^{n-1}}|,|x_{r^{n-2}}|\}$$
$$\prec \max\{|x_{r^{n-1}}|,|x_{r^{n-2}}|\} \prec A\ \ for\,all\ \ n.\ \text{This completes the proof.}$$

Now, we investigate the global attractivity of the equilibrium point $\alpha = 0.$

Lemma 2.1: Assume that $T = \mathbb{Z},\ q \succ p+1,\ and\ m$ is positive rational number and numerator's even positive integers. Furthermore, suppose that

$$R = (q-(p+1))^{1/m}, \tag{2.30}$$

and consider equation (2.3) with the restriction that

$$f : O(0,R)\times O(0,R) \to O(0,R).$$

Let $\{x_n\}$ be a solution of this equation and

$$R_1 = \frac{1+p}{q-(\max\{|x_{-1}|,|x_0|\})^m}. \tag{2.31}$$

Then, $R_1 \in (0,1),\ and$

$$|x_n| \le R_1^{n/2} \times \max\{|x_{-1}|,|x_0|\},\ n = 1,2,\ldots\ . \tag{2.32}$$

Proof: From Theorem 2.5, we have

$$|x_n| \prec A \le R,\ n = -1,0,1,2,\ldots \quad ,$$

where we put n instead of $(n+1)$ in equation (2.3). Then, $(\max\{|x_{-1}|,|x_0|\})^m \prec R^m.$

Thus,

$$q-(\max\{|x_{-1}|,|x_0|\})^m \succ q - R^m = 1+p \succ 0.$$

Hence,

$$0 \prec \frac{1+p}{q-(\max\{|x_{-1}|,|x_0|\})^m} \prec 1.\ \textit{This means that}\ R_1 \in (0,1).$$

Now, we prove that (2.32), by induction on n. From equation (2.3), we have

$$|x_n| \le \frac{1}{|q+x_{n-2}^m|}(p|x_{n-1}|+|x_{n-2}|) \le (\frac{p+1}{|q+x_{n-2}^m|})\max\{|x_{n-1}|,|x_{n-2}|\} \quad (2.33)$$

$$\prec \max\{|x_{n-1}|,|x_{n-2}|\},\ \textit{from}\ (2.26). \quad (2.34)$$

Then, from (2.33), we obtain

$$|x_1| \le (\frac{p+1}{|q+x_{-1}^m|})\max\{|x_0|,|x_{-1}|\}. \quad (2.35)$$

But

$$|q+x_{-1}^m| \ge q-(\max\{|x_0|,|x_{-1}|\})^m \succ q-R^m = 1+p \succ 0.\ \textit{Then},$$
$$|q+x_{-1}^m| \ge q-(\max\{|x_0|,|x_{-1}|\})^m \succ 0.$$

From (2.35), we have

$$|x_1| \le (\frac{p+1}{|q+x_{-1}^m|})\max\{|x_0|,|x_{-1}|\} \le \frac{p+1}{q-(\max\{|x_0|,|x_{-1}|\})^m}\max\{|x_0|,|x_{-1}|\}$$
$$\le R_1 \times \max\{|x_0|,|x_{-1}|\} \prec R_1^{1/2} \times \max\{|x_0|,|x_{-1}|\},$$

and from (2.33) and (2.34), we obtain

$$|x_2| \le (\frac{p+1}{|q+x_0^m|})\max\{|x_1|,|x_0|\} \le \frac{p+1}{q-(\max\{|x_1|,|x_0|\})^m}\max\{|x_1|,|x_0|\}$$
$$\prec \frac{p+1}{q-(\max\{\max\{|x_0|,|x_{-1}|\},|x_0|\})^m} \times \max\{\max\{|x_0|,|x_{-1}|\},|x_0|\}$$
$$\prec \frac{p+1}{q-(\max\{|x_{-1}|,|x_0|\})^m} \times \max\{|x_{-1}|,|x_0|\} \prec R_1 \times \max\{|x_{-1}|,|x_0|\}.$$

Thus, the inequality (2.32) holds for $n=1,2.$ Suppose that the inequality (2.32) holds for $n-1$ and $n-2\,(n\ge 3)$, respectively. By (2.34), we have

$$|x_n| \le \max\{|x_0|,|x_{-1}|\}\ \textit{for all}\ n.$$

So,

$$0 \prec \frac{p+1}{|q+x_{n-2}^m|} \le \frac{p+1}{q-|x_{n-2}|^m} \le \frac{p+1}{q-(\max\{|x_0|,|x_{-1}|\})^m} = R_1.$$

Then, from (2.33), we have

$$|x_n| \le (\frac{p+1}{|q+x_{n-2}^m|})\max\{|x_{n-1}|,|x_{n-2}|\} \le R_1 \max\{|x_{n-1}|,|x_{n-2}|\}$$
$$\le R_1 \times \max\{R_1^{(n-1)/2}, R_1^{(n-2)/2}\} \times \max\{|x_0|,|x_{-1}|\} \le R_1^{n/2}\max\{|x_0|,|x_{-1}|\}.$$

This completes the inductive proof of (2.32).

Lemma 2.2: Assume that $T = h\mathbb{Z} = \{hk : h \in (0,1) \text{ and } k \in \mathbb{Z}\},\ q \succ p+1, \text{ and } m$ is positive rational number and numerator's even positive integers. Furthermore, suppose that (2.30) holds and consider equation (2.3) with the restriction that

$$f : O(0,R) \times O(0,R) \to O(0,R).$$

Let $\{x_{kn} : h \in (0,1) \text{ and } n \in \mathbb{Z}\}$ be a solution of this equation and

$$R_1 = \frac{1+p}{q - (\max\{|x_{-h}|, |x_0|\})^m}. \tag{2.36}$$

$$\text{Then, } R_1 \in (0,1), \text{ and } |x_{hn}| \le R_1^{hn/2} \times \max\{|x_{-h}|, |x_0|\},\ n = 1, 2, \dots . \tag{2.37}$$

Proof: From Theorem 2.5, we have

$$|x_{hn}| \prec A \le R,\ n = -1, 0, 1, \dots \quad ,$$

Where we put hn instead of $h(n+1)$ in equation (2.3). Then, $(\max\{|x_{-h}|, |x_0|\})^m \prec R^m.$ Thus,

$$q - (\max\{|x_{-h}|, |x_0|\})^m \succ q - R^m = 1 + p \succ 0. \text{ Hence,}$$

$$0 \prec \frac{1+p}{q - (\max\{|x_{-h}|, |x_0|\})^m} \prec 1. \textit{ This means that } R_1 \in (0,1).$$

Now, we prove that (2.37), by induction on n. From equation (2.3), we have

$$|x_{hn}| \le \frac{p|x_{h(n-1)}| + |x_{h(n-2)}|}{|q + x_{h(n-2)}^m|} \le \left(\frac{p+1}{|q + x_{h(n-2)}^m|}\right) \max\{|x_{h(n-1)}|, |x_{h(n-2)}|\} \tag{2.38}$$

$$\prec \max\{|x_{h(n-1)}|, |x_{h(n-2)}|\},\ \textit{from } (2.27). \tag{2.39}$$

Then, from (2.38), we obtain

$$|x_h| \le \left(\frac{p+1}{|q + x_{-h}^m|}\right) \max\{|x_0|, |x_{-h}|\}. \tag{2.40}$$

But

$$|q + x_{-h}^m| \ge q - (\max\{|x_0|, |x_{-h}|\})^m \succ q - R^m = 1 + p \succ 0. \textit{ Then, } |q + x_{-h}^m| \ge q - (\max\{|x_0|, |x_{-h}|\})^m \succ 0.$$

From (2.40), we have

$$|x_h| \le \left(\frac{p+1}{|q + x_{-h}^m|}\right) \max\{|x_0|, |x_{-h}|\} \le \frac{p+1}{q - (\max\{|x_0|, |x_{-h}|\})^m} \max\{|x_0|, |x_{-h}|\}$$
$$\le R_1 \times \max\{|x_0|, |x_{-h}|\} \prec R_1^{h/2} \times \max\{|x_0|, |x_{-h}|\},$$

and from (2.38) and (2.39), we obtain

$$
\begin{aligned}
|x_{2h}| &\le (\frac{p+1}{|q+x_0^m|})\max\{|x_h|,|x_0|\} \le \frac{p+1}{q-(\max\{|x_h|,|x_0|\})^m}\max\{|x_h|,|x_0|\} \\
&\prec \frac{p+1}{q-(\max\{\max\{|x_0|,|x_{-h}|\},|x_0|\})^m} \times \max\{\max\{|x_0|,|x_{-h}|\},|x_0|\} \\
&\prec \frac{p+1}{q-(\max\{|x_{-h}|,|x_0|\})^m} \times \max\{|x_{-h}|,|x_0|\} \prec R_1 \times \max\{|x_{-h}|,|x_0|\} \\
&\prec R_1^{2h/2} \times \max\{|x_{-h}|,|x_0|\}.
\end{aligned}
$$

Thus, the inequality (2.37) holds for $n=1,2.$ Suppose that the inequality (2.37) holds for $h(n-1)$ and $h(n-2)\,(n \ge 3)$, respectively. By (2.39), we have

$$|x_n| \le \max\{|x_0|,|x_{-h}|\} \text{ } for\, all \text{ } n.$$

So,

$$0 \prec \frac{p+1}{|q+x_{h(n-2)}^m|} \le \frac{p+1}{q-|x_{h(n-2)}|^m} \le \frac{p+1}{q-(\max\{|x_0|,|x_{-h}|\})^m} = R_1.$$

Then, from (2.38), we have

$$
\begin{aligned}
|x_{hn}| &\le (\frac{p+1}{|q+x_{h(n-2)}^m|})\max\{|x_{h(n-1)}|,|x_{h(n-2)}|\} \le R_1 \max\{|x_{h(n-1)}|,|x_{h(n-2)}|\} \\
&\le R_1^h \max\{|x_{h(n-1)}|,|x_{h(n-2)}|\} \le R_1^h \times \max\{R_1^{h(n-1)/2}, R_1^{h(n-2)/2}\} \times \max\{|x_0|,|x_{-h}|\} \\
&\le R_1^{hn/2} \max\{|x_0|,|x_{-h}|\}.
\end{aligned}
$$

This completes the inductive proof of (2.37).

Lemma 2.3: Assume that $T = \mathbb{N}_0^2 = \{n^2 : n \in \mathbb{N}_0\},\ q \succ p+1,\ and\ m$ is positive rational number and numerator's even positive integers. Furthermore, suppose that (2.30) holds and consider equation (2.3) with the restriction that

$$f : O(0,R) \times O(0,R) \to O(0,R).$$

Let $\{x_k : k = n^2\ and\ n \in \mathbb{N}_0\}$ be a solution of this equation and

$$R_1 = \frac{1+p}{q-(\max\{|x_1|,|x_0|\})^m}. \tag{2.41}$$

Then,

$$|x_k| \le R_1^{k/2} \times \max\{|x_{-1}|,|x_0|\},\ k = 1,4,\dots . \tag{2.42}$$

Proof: From Theorem 2.5, we have

$$|x_k| \prec A \le R,\ k = 1,2,\dots \quad ,$$

where we put n instead of $(n+1)$ in equation (2.3). Then, $(\max\{|x_1|,|x_0|\})^m \prec R^m.$

Thus,

$$q-(\max\{|x_1|,|x_0|\})^m \succ q - R^m = 1+p \succ 0.$$

Hence,

$$0 \prec \frac{1+p}{q-(\max\{|x_1|,|x_0|\})^m} \prec 1. \textit{ This means that } \quad R_1 \in (0,1).$$

Now, we prove that (2.42), by induction on k . From equation (2.3), we have

$$|x_{n^2}| \le \frac{p|x_{(n-1)^2}|+|x_{(n-2)^2}|}{|q+x^m_{(n-2)^2}|}$$

$$\le (\frac{p+1}{|q+x^m_{(n-2)^2}|})\max\{|x_{(n-1)^2}|,|x_{(n-2)^2}|\} \tag{2.43}$$

$$\prec \max\{|x_{(n-1)^2}|,|x_{(n-2)^2}|\}, \textit{ from } (2.28). \tag{2.44}$$

Then, from (2.43), we obtain

$$|x_1| \le (\frac{p+1}{|q+x_1^m|})\max\{|x_0|,|x_1|\}. \tag{2.45}$$

But

$$|q+x_1^m| \ge q-|x_1^m| \ge q-(\max\{|x_0|,|x_1|\})^m \succ q-R^m = 1+p \succ 0. \textit{Then,}$$

$$|q+x_1^m| \ge q-(\max\{|x_0|,|x_1|\})^m \succ 0.$$

From (2.45), we have

$$|x_1| \le (\frac{p+1}{|q+x_1^m|})\max\{|x_0|,|x_1|\} \le \frac{p+1}{q-(\max\{|x_0|,|x_1|\})^m}\max\{|x_0|,|x_1|\}$$
$$\le R_1 \times \max\{|x_0|,|x_1|\} \prec R_1^{1/2} \times \max\{|x_0|,|x_1|\},$$

and from (2.43) and (2.44), we obtain

$$|x_4| \le (\frac{p+1}{|q+x_0^m|})\max\{|x_1|,|x_0|\} \le \frac{p+1}{q-(\max\{|x_1|,|x_0|\})^m}\max\{|x_1|,|x_0|\}$$
$$\prec \frac{p+1}{q-(\max\{\max\{|x_0|,|x_1|\},|x_0|\})^m} \times \max\{R_1\max\{|x_0|,|x_1|\},|x_0|\}$$
$$\prec \frac{p+1}{q-(\max\{|x_1|,|x_0|\})^m} \times R_1 \times \max\{|x_1|,|x_0|\} \prec R_1^2 \times \max\{|x_1|,|x_0|\}$$
$$\prec R_1^{4/2} \times \max\{|x_1|,|x_0|\}.$$

Thus, the inequality (2.42) holds for $n=1,2.$ Suppose that the inequality (2.42) holds for $(n-1)^2$ and $(n-2)^2$, respectively. By (2.44), we have

$$|x_{n^2}| \le \max\{|x_{(n-1)^2}|,|x_{(n-2)^2}|\} \textit{ for all } n \in \mathbb{N}_0.$$

$$So, \quad 0 \prec \frac{p+1}{|q+x^m_{(n-2)^2}|} \leq \frac{p+1}{q-|x_{(n-2)^2}|^m} \leq \frac{p+1}{q-(\max\{|x_0|,|x_1|\})^m} = R_1.$$

Then, from (2.43), we have

$$|x_{n^2}| \leq (\frac{p+1}{|q+x^m_{(n-2)^2}|})\max\{|x_{(n-1)^2}|,|x_{(n-2)^2}|\} \leq R_1 \max\{|x_{(n-1)^2}|,|x_{(n-2)^2}|\}$$
$$\leq \max\{R_1^{(n-1)^2/2}, R_1^{(n-2)^2/2}\} \times \max\{|x_0|,|x_1|\}\} \leq R_1^{(n-2)^2/2} \max\{|x_0|,|x_1|\},\ n=3,4,...$$
$$\leq R_1^{n^2/2} \max\{|x_0|,|x_1|\},\ n=1,2,... \quad .$$

This completes the inductive proof of (2.42).

Lemma 2.4: Assume that $T = r^{\mathbb{N}}, 1 \prec r \leq 2, q \succ p+1, \textit{and } m$ is positive rational number and numerator's even positive integers. Furthermore, suppose that (2.30) holds and consider equation (2.3) with the restriction that

$$f : O(0,R) \times O(0,R) \to O(0,R).$$

Let $\{x_k : k = r^n,\ n \in \mathbb{N} \textit{ and } 1 \prec r \leq 2\}$ be a solution of this equation and

$$R_1 = \frac{1+p}{q-(\max\{|x_{1/r}|,|x_1|\})^m}. \tag{2.46}$$

Then, $R_1 \in (0,1), \textit{ and } |x_k| \leq R_1^{k/2} \times \max\{|x_{1/r}|,|x_1|\}.$ (2.47)

Proof: From Theorem 2.5, we have

$$|x_k| \prec A \leq R,\ k = r^n,\ n \in \mathbb{N}, \textit{ and } 1 \prec r \leq 2,$$

where we put n instead of $(n+1)$ in equation (2.3). Then, $(\max\{|x_{1/r}|,|x_1|\})^m \prec R^m.$ Thus,

$$q-(\max\{|x_{1/r}|,|x_1|\})^m \succ q-R^m = 1+p \succ 0.\ \textit{Hence, } 0 \prec \frac{1+p}{q-(\max\{|x_{1/r}|,|x_1|\})^m} \prec 1.\ \textit{This means that}$$

$$R_1 \in (0,1).$$

Now, we prove that (2.47), by induction on n. From equation (2.3), we have

$$|x_{r^n}| \leq \frac{1}{|q+x^m_{r^{n-2}}|}(p|x_{r^{n-1}}|+|x_{r^{n-2}}|)$$
$$\leq (\frac{p+1}{|q+x^m_{r^{n-2}}|})\max\{|x_{r^{n-1}}|,|x_{r^{n-2}}|\} \tag{2.48}$$
$$\prec \max\{|x_{r^{n-1}}|,|x_{r^{n-2}}|\},\ \textit{from } (2.29). \tag{2.49}$$

Then, from (2.48), we obtain

$$|x_r| \leq (\frac{p+1}{|q+x^m_{1/r}|})\max\{|x_{1/r}|,|x_1|\}. \tag{2.50}$$

But

$$|q+x_{1/r}^m| \geq q-|x_{1/r}^m| \geq q-(\max\{|x_{1/r}|,|x_1|\})^m \succ q-R^m = 1+p \succ 0. \textit{Then},$$

$$|q+x_{1/r}^m| \geq q-|x_{1/r}^m| \geq q-(\max\{|x_{1/r}|,|x_1|\})^m \succ 0.$$

From (2.50), we have

$$|x_r| \leq (\frac{p+1}{|q+x_{1/r}^m|})\max\{|x_{1/r}|,|x_1|\} \leq \frac{p+1}{q-(\max\{|x_{1/r}|,|x_1|\})^m}\max\{|x_{1/r}|,|x_1|\}$$
$$\leq R_1 \times \max\{|x_{1/r}|,|x_1|\} \prec R_1^{r/2} \times \max\{|x_{1/r}|,|x_1|\},$$

and from (2.48) and (2.49), we obtain

$$|x_{r^2}| \leq (\frac{p+1}{|q+x_1^m|})\max\{|x_r|,|x_1|\} \leq \frac{p+1}{q-(\max\{|x_r|,|x_1|\})^m}\max\{|x_r|,|x_1|\}$$
$$\prec \frac{p+1}{q-(\max\{\max\{|x_{1/r}|,|x_1|\},|x_1|\})^m} \times \max\{R_1^{r/2}\max\{|x_{1/r}|,|x_1|\},|x_1|\}$$
$$\prec \frac{p+1}{q-(\max\{|x_{1/r}|,|x_1|\})^m} \times R_1^{r/2} \times \max\{|x_{1/r}|,|x_1|\} \prec R_1 \times R_1^{r/2} \times \max\{|x_{1/r}|,|x_1|\}$$
$$\prec R_1^{2r/2} \times \max\{|x_{1/r}|,|x_1|\} \leq R_1^{r^2/2} \times \max\{|x_{1/r}|,|x_1|\}.$$

Thus, the inequality (2.47) holds for $k = r, r^2.$ Suppose that the inequality (2.47) holds for $k = r^{n-2}$ *and* $k = r^{n-1}, n \in \mathbb{N},$ respectively. By (2.49), we have

$$|x_{r^n}| \leq \max\{|x_{r^{n-1}}|,|x_{r^{n-2}}|\} \textit{ for all } n.$$

So,

$$0 \prec \frac{p+1}{|q+x_{r^{n-2}}^m|} \leq \frac{p+1}{q-|x_{r^{n-2}}|^m} \leq \frac{p+1}{q-(\max\{|x_{1/r}|,|x_1|\})^m} = R_1.$$

Then, from (2.48), we have

$$|x_{r^n}| \leq (\frac{p+1}{|q+x_{r^{n-2}}^m|})\max\{|x_{r^{n-1}}|,|x_{r^{n-2}}|\} \leq R_1 \max\{|x_{r^{n-1}}|,|x_{r^{n-2}}|\}$$
$$\leq \max\{R_1^{r^{n-1}/2}\max\{|x_{1/r}|,|x_1|\}, R_1^{r^{n-2}/2}\max\{|x_{1/r}|,|x_1|\}\}$$
$$\leq R_1^{r^{n-2}/2} \times \max\{|x_{1/r}|,|x_1|\}\}, \; n = 3,4,...$$
$$\leq R_1^{r^n/2} \times \max\{|x_{1/r}|,|x_1|\}, \; n = 1,2,3,... \quad .$$

This completes the inductive proof of (2.47).

Theorem 2.6: Assume that $q \succ p+1,$ *and* m is positive rational number and numerator's even positive integers. Furthermore, suppose that (2.30) holds, and consider equation (2.3) with the restriction that

$$f: O(0,R) \times O(0,R) \to O(0,R).$$

Then the equilibrium point $\alpha = 0$ of the equation (2.3) is a global attractor.

Proof: From Lemmas $2.i,\ i=1,2,3,4,\ x(t)\to\alpha=0\ when\ t\to\infty,$ then the equilibrium point $\alpha=0$ of the equation (2.3) is a global attractor.

Next, we determine the family of invariant intervals centered at $\beta=\sqrt[m]{p-q-1},\ where\ p\succ q+1.$

2.5 Invariant intervals and global attractivity of the nonzero equilibria

In this subsection, we consider the discrete time scales

$T=\mathbb{Z},\ h\mathbb{Z}=\{hk:h\succ 0\ and\ k\in\mathbb{Z}\},\ \ \mathbb{N}_0^2=\{n^2:n\in\mathbb{N}_0\},\ \ and\ r^{\mathbb{N}},\ r\succ 1,\ m=1$. This means that, we determine the family of invariant intervals centered at $\beta=p-q-1.$ To this end, we establish the following relation.

$$x(\sigma(t))-\beta=\frac{px(t)-x(\rho(t))}{q+x(\rho(t))}-\frac{p\beta-\beta}{q+\beta}=(\frac{px(t)+q}{q+x(\rho(t))}-1)-(\frac{p\beta+q}{q+\beta}-1)$$
$$=\frac{px(t)+q}{q+x(\rho(t))}-\frac{p\beta+q}{q+\beta}=(\frac{px(t)+q}{q+x(\rho(t))}-\frac{p\beta+q}{q+x(\rho(t))})+(\frac{p\beta+q}{q+x(\rho(t))}-\frac{p\beta+q}{q+\beta})$$
$$=\frac{p}{q+x(\rho(t))}(x(t)-\beta)+\frac{p\beta+q}{(q+x(\rho(t)))(q+\beta)}(x(\rho(t))-\beta).$$

In view of $p\beta+q=(1+\beta)(q+\beta)=(p-q)(q+\beta),$ the above equality is reduced to

$$x(\sigma(t))-\beta=\frac{p}{q+x(\rho(t))}(x(t)-\beta)+\frac{q-p}{q+x(\rho(t))}(x(\rho(t))-\beta).Thus,$$

$$|x(\sigma(t))-\beta|\le\frac{p}{q+x(\rho(t))}|x(t)-\beta|)+\frac{|q-p|}{q+x(\rho(t))}|x(\rho(t))-\beta|$$
$$\le\frac{p+|p-q|}{q+x(\rho(t))}\max\{|x(\rho(t))-\beta|,|x(t)-\beta|\}. \qquad (2.51)$$

Now, we are ready to describe the family of nested invariant intervals centered at β .

Theorem 2.7: Assume that $p\prec 1/2,\ and\ 3p-1\prec q\prec 1-p.$ Then for every positive number

$$A\le\min\{q-(3p-1),(1-p)-q,1-2p\}, \qquad (2.52)$$

the interval

$$O(\beta,A)=(\beta-A,\beta+A)$$

is invariant for equation (2.3).

Proof: Case (1): If $T=\mathbb{Z}$, given any initial conditions

$$|x_{-1}-\beta|\prec A\ and\ |x_0-\beta|\prec A,$$

we argue that

$$|x_n-\beta|\prec A\ \ for\ all\ n\ by\,induction\,on\ n.$$

The proof is similar to the proof of Theorem 4.2 in [24] and will be omitted.

Case 2: If $T=h\mathbb{Z}$, given any initial conditions

$$|x_{-h}-\beta|\prec A\ and\ |x_0-\beta|\prec A,$$

we argue that

$$| x_{hn} - \beta | \prec A \ \ for\ \ all\ \ n\ by\ induction\ on\ n.$$

It follows from the given initial assertion is true for $n = -1, 0.$ Suppose the asseration is true for $h(n-2)\ and\ h(n-1).$ That is,

$$| x_{h(n-2)} - \beta | \prec A\ and\ | x_{h(n-1)} - \beta | \prec A.$$

Now, we consider x_{hn} where we put hn instead of $h(n+1)$ in equation (2.51). Since,

$$q + x_{h(n-2)} \prec q + (\beta + A) = q + (p - q - 1) + A \prec A - (1-p) \leq A - (1-p) \leq 0,$$

$$we\ have\ \ | q + x_{h(n-2)} | = -(q + x_{h(n-2)}).\ Now,\ the\ condition\ A \leq q - (1-2p)\ is\ equivalent\ to$$

$$-q - (\beta + A) - p = (1-2p) - A \geq 0,$$

$$the\ condition\ A \leq q - (3p-1)\ is\ equivalent\ to$$

$$p - q \leq -(q + (\beta + A) + p),$$

$$and\ the\ condition\ A \leq (1-p) - q\ is\ equivalent\ to$$

$$p - q \geq (q + (\beta + A) + p).$$

$$So,\quad | p - q | \leq -(q + (\beta + A) + p).\ Thus,$$

$$| q + x_{h(n-2)} | - (p + | p - q |) = -(q + x_{h(n-2)}) - p - | p - q |$$

$$\succ -q - (\beta + A) - p - | p - q | \succ -q - (\beta + A) - p + q + (\beta + A) + p \geq 0.$$

From this we deduce that

$$\frac{p + | p - q |}{| q + x_{h(n-2)} |} \prec 1.$$

By (2.51), we have

$$| x_{hn} - \beta | \leq \frac{p + | p - q |}{| q + x_{h(n-2)} |} \times \max\{ | x_{h(n-1)} - \beta |, | x_{h(n-2)} - \beta | \}$$

$$\leq \max\{ | x_{h(n-1)} - \beta |, | x_{h(n-2)} - \beta | \} \prec A.$$

This completes the inductive proof.

When $T = \mathbb{N}_0^2\ and\ r^{\mathbb{N}},\ r \succ 1,$ the proof will be omitted.

Now, we investigate the global attractivity of the equilibrium point $\beta.$

Lemma 2.8: Assume that $T = \mathbb{Z},\ p \prec 1/2,\ and\ 3p - 1 \prec q \prec 1 - p.$ Consider equation (2.3) with the restriction that

$$f : O(\beta, R) \times O(\beta, R) \to O(\beta, R),\ where$$

$$R = \min\{ q - (3p-1), (1-p) - q, 1 - 2p \}. \tag{2.53}$$

Let $\{x_n\}$ be a solution of this equation and

$$M = \frac{p + | p - q |}{1 - p - \max\{ | x_{-1} - \beta |, | x_0 - \beta | \}}. \tag{2.54}$$

Then, $M \in (0,1),\ and$

$$|x_n - \beta| \le M^{n/2} \times \max\{|x_{-1} - \beta|, |x_0 - \beta|\},\ n = 1, 2, \dots \qquad (2.55)$$

The proof is similar to the proof of Lemma 6.1 in [24] and will be omitted.

Lemma 2.9: Assume that $T = h\mathbb{Z} = \{hk : h \in (0,1)\ and\ k \in \mathbb{Z}\},\ p \prec 1/2,\ \ and\ 3p - 1 \prec q \prec 1 - p,$ and (2.53) holds. Consider equation (2.3) with the restriction that

$$f : O(\beta, R) \times O(\beta, R) \to O(\beta, R).$$

Let $\{x_{hn} : h \in (0,1)\ and\, n \in \mathbb{Z}\}$ be a solution of this equation and

$$M = \frac{p + |p - q|}{1 - p - \max\{|x_{-h} - \beta|, |x_0 - \beta|\}}. \qquad (2.56)$$

Then, $M \in (0,1),\ and$

$$|x_{hn} - \beta| \le M^{hn/2} \times \max\{|x_{-h} - \beta|, |x_0 - \beta|\},\ n = 1, 2, \dots (2.57)$$

Proof: Note that

$$1 - p - |x_{-h} - \beta| - (p + |p - q|) \succ 1 - p - R - (p + |p - q|) \succ 1 - 2p - |p - q| - R.$$
$$When,\ p \ge q,$$
$$1 - p - |x_{-h} - \beta| - (p + |p - q|) \succ (q - (3p - 1)) - R \ge 0.$$

$$When,\ p \prec q,\ \ 1 - p - |x_{-h} - \beta| - (p + |p - q|) \succ ((1 - p) - q) - R \ge 0.$$

$$So, \qquad 0 \prec \frac{p + |p - q|}{1 - p - |x_{-h} - \beta|} \prec 1.$$

$$Similarly, \quad 0 \prec \frac{p + |p - q|}{1 - p - |x_{-h} - \beta|} \prec 1.\ Hence,\ M \in (0,1).$$

Next, we prove equation (2.57) by induction on n. By (2.51), where we put hn instead of $h(n+1)$, we have $|x_h - \beta| \le \dfrac{p + |p - q|}{-q - x_{-h}} \times \max\{|x_0 - \beta|, |x_{-h} - \beta|\}$. Since,

$$-q - x_{-h} = (-q - \beta\_ - (x_{-h} - \beta) = 1 - p - (x_{-h} - \beta) \ge 1 - p - \max\{|x_0 - \beta|, |x_{-h} - \beta|\}$$
$$\succ 1 - p - R \succ (1 - 2p) - R \ge 0,\ we\, derive$$
$$M = \frac{p + |p - q|}{1 - p - \max\{|x_0 - \beta|, |x_{-h} - \beta|\}} \ge \frac{p + |p - q|}{-q - x_{-h}} \succ 0.$$
$$Similarly, we\, have$$
$$M \ge \frac{p + |p - q|}{-q - x_0} \succ 0. Then,$$
$$|x_h - \beta| \le M \times \max\{|x_0 - \beta|, |x_{-h} - \beta|\} \le M^{h/2} \times \max\{|x_0 - \beta|, |x_{-h} - \beta|\}.$$

And we have,

$$\begin{aligned} |x_{2h}-\beta| &\le \frac{p+|p-q|}{-q-x_0}\times\max\{|x_0-\beta|,|x_h-\beta|\} \\ &\le M\times\max\{|x_0-\beta|,M\times\max\{|x_0-\beta|,|x_{-h}-\beta|\}\} \\ &\le M\times\max\{|x_0-\beta|,|x_{-h}-\beta|\} \\ &\le M^{2h/2}\times\max\{|x_0-\beta|,|x_{-h}-\beta|\}. \end{aligned}$$

So equation (2.57) holds for $n=1,2$. Suppose that equation (2.57) holds for $h(n-1)$ *and* $h(n\text{-}2)$, respectively. By (2.51) and the inductive hypothesis, we have

$$\begin{aligned} |x_{hn}-\beta| &\le \frac{p+|q-p|}{-q-x_{h(n-2)}}\times\max\{|x_{h(n-1)}-\beta|,|x_{h(n-2)}-\beta|\} \\ &\le \frac{p+|q-p|}{-q-x_{h(n-2)}}\max\{M^{h(n-1)/2},M^{h(n-2)/2}\}\times\max\{|x_0-\beta|,|x_{-h}-\beta|\}\}. \end{aligned}$$

Now, from Theorem (2.7), *we have*

$$|x_{hn}-\beta|\le\max\{|x_{h(n-1)}-\beta|,|x_{h(n-2)}-\beta|\}.$$

By induction on n, we can prove

$$|x_{hn}-\beta|\le\max\{|x_0-\beta|,|x_{-h}-\beta|\},\ \textit{hence,}$$

$$\begin{aligned} -q-x_{h(n-2)} &= (-q-\beta)-(x_{h(n-2)}-\beta)=1-p-(x_{h(n-2)}-\beta) \\ &\ge 1-p-\max\{|x_0-\beta|,|x_{-h}-\beta|\}\succ 1-p-R\succ(1-2p)-R\ge 0, \end{aligned}$$

we conclude $0\prec\dfrac{p+|q-p|}{-q-x_{h(n-2)}}\le M$, *then*

$$\begin{aligned} |x_{hn}-\beta| &\le M\times\max\{M^{h(n-1)/2},M^{h(n-2)/2}\}\times\max\{|x_0-\beta|,|x_{-h}-\beta|\}\} \\ &\le M^{h/2}\times\max\{M^{h(n-1)/2},M^{h(n-2)/2}\}\times\max\{|x_0-\beta|,|x_{-h}-\beta|\}\} \\ &\le M^{nh/2}\times\max\{M^{h/2},1\}\times\max\{|x_0-\beta|,|x_{-h}-\beta|\}\} \\ &\le M^{nh/2}\times\max\{|x_0-\beta|,|x_{-h}-\beta|\}\}. \end{aligned}$$

This completes the inductive proof of equation (2.57).

Lemma 2.10: Assume that $T=\mathbb{N}_0^2=\{n^2:n\in\mathbb{N}_0\},\quad p\prec 1/2, \textit{and } 3p-1\prec q\prec 1-p,$ and (2.53) holds. Consider equation (2.3) with the restriction that

$$f:O(\beta,R)\times O(\beta,R)\to O(\beta,R).$$

Let $\{x_k:k=n^2 \textit{ and } n\in\mathbb{N}_0\}$ be a solution of this equation and

$$M=\frac{p+|p-q|}{1-p-\max\{|x_1-\beta|,|x_0-\beta|\}}. \tag{2.58}$$

Then, $M\in(0,1)$ *and*

$$|x_k-\beta|\le M^{k/2}\times\max\{|x_1-\beta|,|x_0-\beta|\},\ k=1,4,\dots. \tag{2.59}$$

Lemma 2.11: Assume that $T = r^{\mathbb{N}}, 1 \prec r \leq 2, \quad p \prec 1/2, and \ 3p-1 \prec q \prec 1-p,$ and (2.53) holds. Consider equation (2.3) with the restriction that

$$f : O(\beta, R) \times O(\beta, R) \to O(\beta, R).$$

Let $\{x_k : k = r^n, 1 \prec r \leq 2, \ and \ n \in \mathbb{N}\}$ be a solution of this equation and

$$M = \frac{p + |p-q|}{1 - p - \max\{|x_{1/r} - \beta|, |x_1 - \beta|\}}. \qquad (2.60)$$

Then, $M \in (0,1)$ *and*

$$|x_k - \beta| \leq M^{k/2} \times \max\{|x_{1/r} - \beta|, |x_1 - \beta|\}. \qquad (2.61)$$

Theorem 2.9: Assume that $p \prec 1/2, and \ 3p-1 \prec q \prec 1-p,$ and (2.53) holds. Consider equation (2.3) with the restriction that

$$f : O(\beta, R) \times O(\beta, R) \to O(\beta, R).$$

Then the equilibrium point β of the equation (2.3) is a global attractor.

Proof: From Lemmas $2.i, \ i = 8,9,10,11,$ *we obtain* $x(t) \to \beta$ *when* $t \to \infty,$ then the equilibrium point β of the equation (2.3) is a global attractor.

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