Generalized ordered weighted power utility averaging operator and its applications to group decision-making

Jianwei Gao[1], Huihui Liu[2]

School of Economics and management, North China Electric Power University, Beijing, 102206, China
Email: liuhooi@sina.com

Abstract— *This paper develops a new operator called the generalized ordered weighted power utility averaging (GOWPUA) operator, which first introduces the risk attitude of decision makers (DMs) in the aggregation process. We study its properties and families. To determine the GOWPUA operator weights, we put forward an orness measure of the GOWPUA operator and analyze its properties. Considering that different DMs may have different perspectives towards decision-making, which can be characterized by different degrees of orness, we construct a new nonlinear optimization model to determine the optimal weights which can aggregate all the individual sets of weights into an overall set of weights. Finally, based on the GOWPUA operator, a method for multiple attribute group decision-making (MAGDM) is developed.*

Keywords— Multiple attribute group decision-making, Aggregation operator, Utility function, Orness.

I. INTRODUCTION

Multiple attribute group decision-making (MAGDM) considers the problem of selecting alternatives that are associated with incommensurate and conflicting attributes by a cooperative group, known as the group decision-making [1]. To choose a desirable alternative, decision makers (DMs) often present their preference information which needs to be aggregated via some proper approaches. There are many methods for aggregating the information [2-17]. One of the most popular methods for aggregating decision formation is the ordered weighted averaging (OWA) operator developed by Yager [14]. It provides a general class of parametric aggregation operators and has been shown to be useful for modeling many different kinds of aggregation problems. Up to now, OWA operator has been used in a wide range of applications [7-10, 16, 18].

Motivated by the OWA operator, an extension to the OWA operator is the generalized OWA (GOWA) operator, which combined the OWA operator with the generalized mean operator [15]. It generalized a wide range of aggregation operators such as the OWA operator [14], the ordered weighted geometric averaging (OWGA) operator [19], the ordered weighted harmonic averaging (OWHA) operator [15], etc. Based on the optimization theory, Zhou and Chen [20] presented the generalized ordered weighted logarithm averaging (GOWLA) operator, which is an extension of the OWGA operator. Other extension of the OWA operator can be founded in literature [6, 21]. However, the above aggregation operators only focused on using the mean to eliminate the difference, and did not consider the DMs' risk attitude in the aggregation process.

Another important issue of applying the OWA operator for MAGDM is how to determine the associated weights. Many researchers have focused on this issue, and developed some useful approaches to obtaining the OWA weights. For example, O'Hagan [22] suggested a maximum entropy approach for obtaining the OWA operator weights for a given level of orness. Fullér and Majlender [4] proposed an analytic approach for obtaining maximal entropy OWA operator weights for a given orness level. Wang and Parkan [23] proposed a minimax disparity approach for obtaining OWA operator weights for a given orness level. Majlender [24] developed a maximal Rényi entropy method for generating a parametric class of OWA operators and the maximal Rényi entropy OWA weights. Other extension approaches to determining the OWA operator weights can be founded in literature [6, 21, 25, 26]. The methods mentioned above assume that any individual weight vector is equal to the optimal weight vector and correspondingly, and there is only one degree of orness to characterize the DMs' attitude towards decision-making. As a result, there is only one set of OWA operator weights to be generated. However, this is not consistent with the real situation. In fact, multiple DMs may join in decision-making process to reach a holistic opinion that reflects all the participants' collective view, and different DMs may have different degrees of orness, which leads to the corresponding OWA operator weights may also be different. So it is necessary to introduce a new method to aggregate all the participants' preference in MAGDM.

This paper aims to develop a new class of aggregation operator based on power utility function, which incorporates the risk attitude of DMs in the aggregation process. First, based on an optimal deviation model, we provide a new operator called the generalized ordered weighted power utility averaging (GOWPUA) operator, which is an extension of the GOWLA operator presented by Zhou and Chen [20]. Then we study some properties of the GOWPUA operator and prove that it is commutative, monotonic, bounded and idempotent. Furthermore, we investigate the families of the GOWPUA operator, and find that they include a wide range of aggregation operators such as the OWLGA operator [20], OWGA operator [19], OWLHA operator, GOWLA operator [20], OWA operator [14], etc. The main advantage of the GOWPUA operator is that it can not only reflect the DMs' risk attitude towards the aggregation information, but also provide a very general formulation that includes a wide range of aggregation operators.

To determine the GOWPUA operator weights, we present an orness measure of the GOWPUA operator, which is an extension of the OWA operator orness. We further discuss some properties associated with this orness measure. Noting that different DMs may have different perspectives towards decision-making, which can be characterized by different degrees of orness, this situation leads to different sets of the GOWPUA operator weights corresponding to different orness degrees. We then construct a new nonlinear optimization model to determine the optimal weight vector of the GOWPUA operator which aggregates all the individual sets of weights into an overall set of weights. The main advantage of the nonlinear model is that it can not only minimize the differences between the degrees of orness provided by each DM and the degree of orness corresponding to an optimal aggregated weight vector, but also produce as equally important weights as possible.

Furthermore, based on the GOWPUA operator, a new approach for MAGDM is developed. This approach is also applicable to different group decision-making problems effectively such as human resource management, engineering management and financial management, etc.

The rest of the paper is organized as follows. Section 2 presents the GOWPUA operator and analyzes its properties. In addition, the families of the GOWPUA operator are investigated. Section 3 proposes an orness measure of the GOWPUA operator and discusses its properties. We further provide a new nonlinear model for determining the optimal weights which can aggregate each DM's opinion. Section 4 develops an approach for MAGDM under the GOWPUA operator and the conclusions are drawn in Section 5.

II. GENERALIZED ORDERED WEIGHTED POWER UTILITY AVERAGING OPERATOR

In general, the basic feature of aggregation operators is the non-decreasing monotonicity, expressing the idea that "an increase of any of the input values cannot decrease the output value". The desirable properties of each aggregation operator are commutative, monotonic, bounded and idempotent. Each aggregation operator mentioned above satisfies these properties. Nevertheless, the aforementioned aggregation operator in fail to capture the DM's psychological characters in the aggregation process.

Therefore, when aggregating the input arguments, we attempt to partially fill this gap by introducing the utility function in the aggregation process. The utility function not only satisfies the basic feature and desirable properties of aggregation operators, but also can reflect the risk attitude of the DMs towards the input argument information. Noting that Zhou and Chen [20] utilized the logarithm function to derive the GOWLA operator, we will focus on the power utility function, which is an extension of the logarithm function, to develop the aggregation operator.

A utility function $u(x)$ is a non-decreasing real valued function defined on the real numbers, which just captures the idea of aggregation operator that an increase of any of the input values cannot decrease the output value. We investigate the power utility function: $u(x)=(x^{\gamma}-1)/\gamma$, where the risk aversion coefficient γ satisfies $\gamma\in(-\infty,0)\cup(0,1)$. Especially, when $\gamma\to 0$, then the power utility function will degenerate to the logarithm function: $u_L(x)=\ln x$.

Pratt [27] and Arrow [28] suggested the relative risk aversion function, $r(x)=-x\cdot u''(x)/u'(x)$, which is called the Pratt-Arrow measure of relative risk aversion. Note that the relative risk aversion of the power utility is $1-\gamma$, meaning that with the decrease of the risk aversion coefficient γ, the relative risk aversion will increase, and consequently the risk attitude of DM's involved in the evaluation of decision information becomes more prudent.

2.1 GOWPUA operator

Let $\mathbf{x}=(x_1,x_2,\cdots,x_n)$ be a collection of arguments and $\mathbf{w}=(w_1,w_2,\cdots,w_n)$ be a weight vector such that $w_i\in[0,1]$ and $\sum_{i=1}^{n} w_i=1$. Note that the power utility $u(x)\geq 0$, which implies that $x_i\geq 1$ for all *i*. For simplicity, similar to the assumption of Zhou and

Chen [20], this paper assumes that $x_i \geq 1$ for all i and $\Omega = \{x | x \geq 1, x \in R^+\}$, and that the utility aggregation operator of dimension n is a mapping f determined by:

$$u(y) = f(u(x_1), u(x_2), \cdots, u(x_n)).$$

In the aggregation process, we hope that the smaller the deviation between the utility values $u(x_i)$ $(i = 1, 2, \cdots, n)$ and the aggregation result $u(y)$ is, the better effect of aggregation method shows. Hence, to minimize the deviation between $u(y)$ and $u(x_i)(i = 1, 2, \cdots, n)$, we have

$$\min \sum_{i=1}^{n} w_i \left[\left((x_i^\gamma - 1)/\gamma \right)^\lambda - \left((y^\gamma - 1)/\gamma \right)^\lambda \right]^2, \quad (1)$$

where λ is a parameter such that $\lambda \in (-\infty, 0) \cup (0, +\infty)$. First-order condition (matching the derivative with respect to y to 0) of the objective function in Expression (1) yields that

$$y = \left(\left(\sum_{i=1}^{n} w_i (x_i^\gamma - 1)^\lambda \right)^{1/\lambda} + 1 \right)^{1/\gamma}. \quad (2)$$

Based on Eq. (2), we propose a generalized ordered weighted power utility averaging operator.

Definition 1. A generalized ordered weighted power utility averaging (GOWPUA) operator of dimension n is a mapping GOWPUA: $R^n \to R$ such that

$$GOWPUA(\mathbf{x}) = \left(\left(\sum_{i=1}^{n} w_i (z_i^\gamma - 1)^\lambda \right)^{1/\lambda} + 1 \right)^{1/\gamma}, \quad (3)$$

where the weight vector $\mathbf{w} = (w_1, w_2, \cdots, w_n)$ satisfies $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, and z_i is the ith largest of x_j, and the parameter λ satisfies $\lambda \in (-\infty, 0) \cup (0, +\infty)$, and the risk aversion coefficient γ satisfies $\gamma \in (-\infty, 0) \cup (0, 1)$.

The following proposition shows that the GOWPUA operator is monotonic, bounded, commutative, idempotent, thus satisfying common properties of aggregation operators [15].

Proposition 1 (*Properties of GOWPUA*). The GOWPUA operator given in Definition 1 satisfies the

following properties:

(i) (*Idempotency*). If $x_i = x$ for $i = 1, 2, \cdots, n$, then $GOWPUA(\mathbf{x}) = x$.

(ii) (*Monotonicity*). If $a_i \geq b_i$ for $i = 1, 2, \cdots, n$, then $GOWPUA(a_1, a_2, \cdots, a_n) \geq GOWPUA(b_1, b_2, \cdots, b_n)$.

(iii) (*Boundedness*). If $z_1 = \max_i \{x_i\}$ and $z_n = \min_i \{x_i\}$, then $z_n \leq GOWPUA(\mathbf{x}) \leq z_1$.

(iv) (*Commutativity*). If $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ is any permutation of the arguments $\mathbf{x} = (x_1, x_2, \cdots, x_n)$, then we have that $GOWPUA(\mathbf{x}) = GOWPUA(\mathbf{v})$.

Proof. (i) If $x_i = x$ for all i, according to Eq. (10), we derive $GOWPUA(\mathbf{x}) = \left(\left(\sum_{i=1}^{n} w_i (x^\gamma - 1)^\lambda \right)^{1/\lambda} + 1 \right)^{1/\gamma} = x$.

(ii) For simplicity, let f denote the GOWPUA operator. According to the above equation, by taking the first partial derivative of f with respect to z_i, we have that

$$\frac{\partial f}{\partial z_i} = \left(\left(\sum_{i=1}^{n} w_i (z_i^\gamma - 1)^\lambda \right)^{1/\lambda} + 1 \right)^{1/\gamma - 1} \cdot \left(\sum_{i=1}^{n} w_i (z_i^\gamma - 1)^\lambda \right)^{1/\lambda - 1} \cdot w_i z_i^{\gamma - 1} (z_i^\gamma - 1)^{\lambda - 1} \geq 0.$$

Thus, we conclude that f monotonically increase with respect to z_i . Considering that $a_i \ge b_i$ for all i, we get $GOWPUA(a_1,a_2,\cdots,a_n) \ge GOWPUA(b_1,b_2,\cdots,b_n)$.

(iii) If $z_1 = \max_i \{x_i\}$, then by Property 2, we have that $GOWPUA(\mathbf{x}) \le GOWPUA(z_1,z_1,\cdots,z_1) = z_1$ .By the same token, for $z_n = \min_i \{x_i\}$, we can conclude that $GOWPUA(\mathbf{x}) \ge z_n$. Therefore, $z_n \le GOWPUA(\mathbf{x}) \le z_1$.

(iv) Let $GOWPUA(\mathbf{x}) = \left(\left(\sum_{i=1}^{n} w_i (z_i^{\gamma}-1)^{\lambda}\right)^{1/\lambda}+1\right)^{1/\gamma}$ and $GOWPUA(\mathbf{v}) = \left(\left(\sum_{i=1}^{n} w_i (k_i^{\gamma}-1)^{\lambda}\right)^{1/\lambda}+1\right)^{1/\gamma}$, where z_i and k_i are respectively i$^{\text{th}}$ largest of x_j and v_j $(j=1,2,\cdots,n)$. Since $(v_1,v_2,\cdots,v_n)$ is any permutation of the arguments $(x_1,x_2,\cdots,x_n)$, we can obtain that $z_i = k_i$ for all i. So we obtain that $GOWPUA(\mathbf{x}) = GOWPUA(\mathbf{v})$.

2.2 Families of GOWPUA operator

Taking special values of the parameters λ,γ and the weight vector $\mathbf{w}$, the GOWPUA operator degenerates into many different aggregation operators including the OWGA operator [19], GOWLA operator [20], OWA operator [14], etc.

Theorem 1. If $\gamma \to 0$, then we have

$$\lim_{\gamma\to 0} GOWPUA(\mathbf{x}) = \exp\left(\left(\sum_{i=1}^{n} w_i (\ln z_i)^{\lambda}\right)^{1/\lambda}\right).$$

Proof. Let $G(\mathbf{x})$ denote the GOWPUA operator. By taking the natural logarithm of f and using the L'Hôpital's rule, we have

$$\lim_{\gamma\to 0} \ln G(\mathbf{x}) = \lim_{\gamma\to 0} \frac{1}{\gamma} \ln\left(\gamma\left(\sum_{i=1}^{n} w_i \left((z_i^{\gamma}-1)/\gamma\right)^{\lambda}\right)^{1/\lambda}+1\right) = \left(\sum_{i=1}^{n} w_i (\ln z_i)^{\lambda}\right)^{1/\lambda}.$$

Thus, $\lim_{\gamma\to 0} G(\mathbf{x}) = \exp\left(\left(\sum_{i=1}^{n} w_i (\ln z_i)^{\lambda}\right)^{1/\lambda}\right)$.

Remark 1. Theorem 1 is just the GOWLA operator [20], meaning that the GOWPUA operator is an extension of the GOWLA operator [20].

Corollary 1. When $\gamma \to 0$, by choosing different parameters of λ , we can derive the following aggregation operators.

(1) If $\lambda \to 0$, then the GOWPUA operator reduces to the OWLGA operator [20]: $\lim_{\gamma\to 0,\,\lambda\to 0} G(\mathbf{x}) = \exp\left(\prod_{i=1}^{n} (\ln z_i)^{w_i}\right)$.

(2) If $\lambda = 1$, then the GOWPUA operator degenerates to the OWGA operator [15]: $\lim_{\gamma\to 0,\,\lambda=1} G(\mathbf{x}) = \prod_{i=1}^{n} z_i^{w_i}$.

(3) If $\lambda = -1$, then the GOWPUA operator becomes: $\lim_{\gamma\to 0,\,\lambda=-1} G(\mathbf{x}) = \exp\left(1\Big/\sum_{i=1}^{n} w_i/\ln z_i\right)$.

Proof. Based on Theorem 1, we derive $\lim_{\gamma\to 0,\,\lambda\to 0} G(\mathbf{x}) = \lim_{\lambda\to 0} \exp\left(\left(\sum_{i=1}^{n} w_i (\ln z_i)^{\lambda}\right)^{1/\lambda}\right)$. By the L'Hôpital's rule, we have

$$\lim_{\gamma\to 0,\,\lambda\to 0} G(\mathbf{x}) = \exp\left[\exp\left(\lim_{\lambda\to 0} \frac{1}{\lambda} \ln\left(\sum_{i=1}^{n} w_i (nz_i)^{\lambda}\right)\right)\right] = \exp\left(\prod_{i=1}^{n} (\ln z_i)^{w_i}\right).$$

According to Theorem 1, it is easy to derive the OWGA operator [15] and the ordered generalized weighted logarithm harmonic averaging (OWLHA) operator. □

Theorem 2. If $\gamma \to 1$, then we get $\lim_{\gamma\to 1} G(\mathbf{x}) = \left(\sum_{i=1}^{n} w_i (z_i - 1)^{\lambda}\right)^{1/\lambda} + 1$.

Proof. Based on Eq. (3), it is easy to find the conclusion. □

Remark 2. Theorem 2 is similar to the GOWA operator [15]. Note that if $\gamma \to 1$, then the power utility will become $u(x) = x - 1$, which is just a linear function. In this case, if the utility function meaning is neglected, Theorem 2 can be explained by the fact that when reordering the arguments in descending order, DMs first subtract the constant one for all the input arguments, after aggregating the arguments, they should add the constant one so as to maintain the output values unchanged.

Corollary 2. When $\gamma \to 1$, by choosing different parameters of λ, we can derive the following aggregation operators.

(1) If $\lambda \to 0$,then we have $\lim_{\gamma\to1,\,\lambda\to0} G(\mathbf{x}) = \prod_{i=1}^{n}(z_i-1)^{w_i}+1$, which has a similar form of the OWGA operator [19].

(2) If $\lambda = 1$, then the GOWPUA operator degenerates to the OWA operator [14].

(3) If $\lambda = -1$, then we get $\lim_{\gamma\to1,\,\lambda=-1} G(\mathbf{x}) = \left(1\Big/\sum_{i=1}^{n} w_i/(z_i-1)\right)+1$, which has a similar form of the OWHA operator.

Proof. According to Theorem 2, by the L'Hôpital's rule, we derive that

$$\lim_{\gamma\to1,\,\lambda\to0} G(\mathbf{x}) = \lim_{\lambda\to0}\exp\left[\frac{1}{\lambda}\left(\ln\left(\sum_{i=1}^{n} w_i(z_i-1)^{\lambda}\right)\right)\right]+1 = \exp\left(\sum_{i=1}^{n} w_i \ln(z_i-1)\right)+1 = \prod_{i=1}^{n}(z_i-1)^{w_i}+1 .$$

Thus, $\lim_{\gamma\to1,\,\lambda\to0} G(\mathbf{x}) = \prod_{i=1}^{n}(z_i-1)^{w_i}+1$. Based on Theorem 2, it is easy to obtain OWA operator and similar form of the OWHA operator.

Remark 3. According to Eq. (3), we note that the GOWPUA operator reduces to zero for $\gamma \to -\infty$. This situation can be understood by the meaning of the risk aversion coefficient γ. Noting that when aggregating the input arguments, the DM adds his/her risk judgment in these arguments. With the decrease of the risk aversion coefficient γ, the risk attitude of DM's involved in the judgment of input arguments becomes more prudent and consequently, in the limit $\gamma \to -\infty$, no aggregation takes place.

Theorem 3. As $\lambda \to +\infty$, we have the following statements.

(1) If $0 < \gamma < 1$ (or $\gamma \to 0$, $\gamma \to 1$),then $G(\mathbf{x}) = z_1$ (i.e., maximum operator).

(2) If $\gamma < 0$, then $G(\mathbf{x}) = z_n$ (i.e., minimum operator).

Proof. According to Proposition 1, we derive that $GOWPUA(x_1, x_2, \cdots, x_n) \le GOWPUA(z_1, z_1, \cdots, z_1) = z_1$, where $z_1 = \max_i\{x_i\}$. When $\lambda \to +\infty$, by choosing $0 < \gamma < 1$ (or $\gamma \to 0$, $\gamma \to 1$)and $\gamma < 0$, we will obtain the opposite results in the following two cases:

(1) I. If $0 < \gamma < 1$ and $\lambda \to +\infty$, then $\left(\left(\sum_{i=1}^{n} w_i(z_i^{\gamma}-1)^{\lambda}\right)^{1/\lambda}+1\right)^{1/\gamma} \ge \left(\left(w_1(z_1^{\gamma}-1)^{\lambda}\right)^{1/\lambda}+1\right)^{1/\gamma} = w_1^{1/\gamma\lambda} z_1$. According to the above inequality, by taking the limitation on both sides, we get $\lim_{\lambda\to+\infty}\left(\left(\sum_{i=1}^{n} w_i(z_i^{\gamma}-1)^{\lambda}\right)^{1/\lambda}+1\right)^{1/\gamma} \ge \lim_{\lambda\to+\infty} w_1^{1/\gamma\lambda} z_1 = z_1.$

Therefore, we find that $\lim_{\lambda\to+\infty} G(\mathbf{x}) = z_1$, which is just the maximum operator.

II. If $\gamma \to 0$, based on Theorem 1, we have that $\lim_{\gamma\to0} G(\mathbf{x}) = \exp\left(\left(\sum_{i=1}^{n} w_i(\ln z_i)^{\lambda}\right)^{1/\lambda}\right) \le \exp\left(\left(\sum_{i=1}^{n} w_i(\ln z_1)^{\lambda}\right)^{1/\lambda}\right) = z_1$.

When $\lambda \to +\infty$, we get $\lim_{\gamma\to0,\,\lambda\to+\infty} G(\mathbf{x}) = \lim_{\lambda\to+\infty}\exp\left(\left(\sum_{i=1}^{n} w_i(\ln z_i)^{\lambda}\right)^{1/\lambda}\right) \ge \lim_{\lambda\to+\infty}\exp\left(\left(w_1(\ln z_1)^{\lambda}\right)^{1/\lambda}\right) = z_1$. Thus, we derive that $\lim_{\gamma\to0,\,\lambda\to+\infty} G(\mathbf{x}) = z_1$, which is just the maximum operator.

III. If $\gamma \to 1$, according to Theorem 2, similar to the proof II, we can obtain the conclusion.

(2) If $\gamma < 0$ and $\lambda \to +\infty$, we have that $\lambda\gamma \to -\infty$. Similar to the proof I, we can get the minimum operator.

Theorem 4. When $\lambda \to -\infty$, we can obtain the following statements.

(1) If $0 < \gamma < 1$, then $G(\mathbf{x}) = z_n$ (i.e., minimum operator).

(2) If $0 < \gamma < 1$ (or $\gamma \to 0$, $\gamma \to 1$), then $G(\mathbf{x}) = z_1$ (i.e., maximum operator).

Proof. Similar to the proof of Theorem 3, it is easy to obtain the conclusion. □

Theorem 5. If $\lambda \to 0$, we have that $\lim_{\lambda \to 0} G(\mathbf{x}) = \left(\prod_{i=1}^{n} \left(z_i^{\gamma} - 1 \right)^{w_i} + 1 \right)^{1/\gamma}$, which is called the ordered weighted power utility geometric averaging (OWPUGA) operator.

Proof. By the L'Hôpital's rule, we obtain that $\lim_{\lambda \to 0} G(\mathbf{x}) = \left[\exp\left(\lim_{\lambda \to 0} \frac{1}{\lambda} \ln\left(\sum_{i=1}^{n} w_i (z_i^{\gamma} - 1)^{\lambda} \right) \right) + 1 \right]^{1/\gamma} = \left(\prod_{i=1}^{n} \left(z_i^{\gamma} - 1 \right)^{w_i} + 1 \right)^{1/\gamma}$. Thus, $\lim_{\lambda \to 0} G(\mathbf{x}) = \left(\prod_{i=1}^{n} \left(z_i^{\gamma} - 1 \right)^{w_i} + 1 \right)^{1/\gamma}$. □

Theorem 6. If $\lambda = 1$, we have that $G(\mathbf{x}) = \left(\sum_{i=1}^{n} w_i z_i^{\gamma} \right)^{1/\gamma}$, which is named as the ordered weighted power utility averaging (OWPUA) operator.

Proof. According to Eq. (3), it is easy to find the conclusion. □

Corollary 3. If $\lambda = 1$ and $\gamma = -1$, then the GOWPUA operator will degenerate to the OWHA operator.

Proof. According to Theorem 6, it is easy to find the conclusion. □

Remark 4. If we consider the possible values of the weight vector $\mathbf{w} = (w_1, w_2, \cdots, w_n)^T$ in the GOWPUA operator, a group of particular cases can be obtained as follows.

- The maximum operator is derived if $w_1 = 1$ and $w_i = 0 (i \neq 1)$.
- The minimum operator is founded if $w_n = 1$ and $w_i = 0 (i \neq n)$.
- The Step-GOWPUA operator is formed if $w_k = 1$ and $w_i = 0 (i \neq k)$ and z_k is the largest of x_i, where $Step-GOWPUA(\mathbf{x}) = z_k$.
- The generalized power utility averaging (GPUA) operator is derived if $w_i = \frac{1}{n}$, where $GPUA(\mathbf{x}) = \left(\left(\sum_{i=1}^{n} \frac{1}{n} (x_i^{\gamma} - 1)^{\lambda} \right)^{1/\lambda} + 1 \right)^{1/\gamma}$.
- The Windows-GOWPUA operator is derived if $w_i = \frac{1}{m} (k \le i \le k + m - 1)$ and $w_i = 0 \quad (i < k \quad and\ i \ge k + m)$, where $Windows - GOWPUA(\mathbf{x}) = \left(\left(\frac{1}{m} \sum_{i=k}^{k+m-1} (z_i^{\gamma} - 1)^{\lambda} \right)^{1/\lambda} + 1 \right)^{1/\gamma}$.

Especially, if $k = p = 1$, then the Window-GOWPUA operator reduces to the maximum operator. If $p = 1$ and $k = n$, then the Window-GOWPUA operator degenerates to the minimum operator. If $p = n$ and $k = 1$, then it becomes the GPUA operator.

- The Olympic-GOWPUA operator is obtained if $w_i = \frac{1}{n-2} (2 \le i \le n - 1)$ and $w_i = 0 (i = 1, n)$, where $Olympic - GOWPUA(\mathbf{x}) = \left(\left(\frac{1}{n-2} \sum_{i=2}^{n-1} (z_i^{\gamma} - 1)^{\lambda} \right)^{1/\lambda} + 1 \right)^{1/\gamma}$.

III. A MODEL FOR DETERMINING THE GOWPUA OPERATOR WEIGHTS

To determine the weights of the GOWPUA operator, we propose an orness measure of the GOWPUA operator and analyze its properties. We further propose a new optimization method, which can aggregate DMs' opinion and obtain an optimal aggregated weight vector for the GOWPUA operator.

3.1 An orness measure for the GOWPUA operator

The orness measure of the OWA operator was presented by Yager [14], which is defined as follows.

Definition 2. Assuming that *orness* is an OWA aggregation operator with weight vector $\mathbf{w}=(w_1,w_2,\cdots,w_n)$, the degree of "orness" associated with this operator is defined as:

$$orness(\mathbf{w})=\frac{1}{n-1}\sum_{i=1}^{n}\left[(n-i)w_i\right].$$

It can be shown that when $\mathbf{w}=(1,0,\cdots,0)$, $orness(\mathbf{w})=1$; when $\mathbf{w}=(0,0,\cdots,1)$, $orness(\mathbf{w})=0$; when $\mathbf{w}=(1/n,1/n,\cdots,1/n)$, $orness(\mathbf{w})=1/2$.

The orness measure is also called the attitudinal character of the aggregation, which can be regarded as the OWA aggregation of the arguments $x_i=(n-i)/(n-1)$ for all i. By using this method, Yager [15] presented the orness measure of the GOWA operator:

$$orness(\mathbf{w})=\left[\sum_{i=1}^{n}w_i\left((n-i)/(n-1)\right)^{\lambda}\right]^{1/\lambda},$$

where λ is a parameter such that $\lambda\in(-\infty,0)\cup(0,+\infty)$. If $\lambda=1$, then the orness measure of the GOWA operator will degenerate to the orness measure of the OWA operator.

Following Yager [15], we can define an orness measure of the GOWPUA operator as follows:

Definition 3. The degree of orness associated with the GOWPUA operator is defined as follows:

$$orness(\mathbf{w})=\left[\left(\sum_{i=1}^{n}w_i\left(\left(\frac{n-i}{n-1}\right)^{\gamma}-1\right)^{\lambda}\right)^{1/\lambda}+1\right]^{1/\gamma},$$

where λ is a parameter such that $\lambda\in(-\infty,0)\cup(0,+\infty)$ and $\gamma\in(-\infty,0)\cup(0,1)$.

Remark 5. It can be shown that for the GOWPUA operator: when $\mathbf{w}=(1,0,\cdots,0)$, $orness(\mathbf{w})=1$; when $\mathbf{w}=(0,0,\cdots,1)$, $orness(\mathbf{w})=0$. In particular, if $\gamma\to 0$, then the orness measure of the GOWPUA operator will degenerate to the case of GOWLA operator [20].

Theorem 7. The orness measure of the GOWPUA operator satisfies $0\le orness(\mathbf{w})\le 1$.

Proof. According to Proposition 1, we derive that $\left[\left(\sum_{i=1}^{n}w_i\left(\left(\frac{n-n}{n-1}\right)^{\gamma}-1\right)^{\lambda}\right)^{1/\lambda}+1\right]^{1/\gamma}\le orness(\mathbf{w})\le\left[\left(\sum_{i=1}^{n}w_i\left(\left(\frac{n-1}{n-1}\right)^{\gamma}-1\right)^{\lambda}\right)^{1/\lambda}+1\right]^{1/\gamma}$.

That is, $0\le orness(\mathbf{w})\le 1$. □

3.2 An optimization model for determining GOWPUA operator weights under the orness measure

Based on the orness measure and the dispersion measure, O'Hagan [22]developed a maximum entropy approach to determining the OWA operator weights. Fullér and Majlender [5]provided a minimum variance method, which demands the solution to the quadratic programming problem for minimizing the variance of the OWA operator weights under a given degree of orness. Wang and Parkan [23]proposed a model for minimizing the maximum disparity between two adjacent weights under a given level of orness.

The above three approaches all imply the use of more information from all attributes. As Wang and Parkan [29] pointed out, except for computational simplicity or complexity, there are no significant differences among the three alternative approaches.

Considering the importance of the OWA weights, Wang et al. [30] constructed the chi-square (χ^2) model for determining the OWA operator weights, which can be expressed as follows:

$$\begin{aligned} &Minimize \quad \sum_{i=1}^{n-1}\left(\frac{w_i}{w_{i+1}}+\frac{w_{i+1}}{w_i}-2\right), 0 \le \alpha \le 1, \\ &\text{s.}t. \quad orness(\mathbf{w}) = \alpha = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i \\ &\sum_{i=1}^{n} w_i = 1,\ 0 \le w_i \le 1, \quad i = 1,\cdots,n. \end{aligned} \tag{4}$$

However, these approaches mentioned above suppose that any individual weight vector is equal to the optimal weight vector and correspondingly, there is only one degree of orness to characterize the DMs' attitude towards decision-making, and only one set of OWA operator weights to be generated. This situation is not consistent with the reality. In fact, different DMs may have different perspectives, which can be characterized by different degrees of orness. As a result, there exist different sets of OWA operator weights corresponding to different orness degrees. Hence, it is necessary to develop a new method to aggregate all the individual sets of weights into an overall set of weights.

In order to generate such an optimal weight vector for the GOWPUA operator in MAGDM, we propose a new method for determining the weights. Let α_k be the degree of orness provided by the kth DM $(k = 1,\cdots,l)$, and $\mathbf{w}^* = \left(w_1^*, w_2^*, \cdots, w_n^*\right)$ an optimal aggregated weight vector for the GOWPUA operator. The degree of orness corresponding to such an optimal aggregated weight vector $\mathbf{w}^* = \left(w_1^*, w_2^*, \cdots, w_n^*\right)$ is generally not equal to the $\alpha_k \ \ (k = 1,\cdots,l)$ provided by the DMs. In other words,

$$orness(\mathbf{w}^*) = \left[\left(\sum_{i=1}^{n} w_i^*\left(\left(\frac{n-i}{n-1}\right)^{\gamma}-1\right)^{\lambda}\right)^{1/\lambda}+1\right]^{1/\gamma} \ne a_k, (k = 1,\cdots,l).$$

To measure the differences between $orness(\mathbf{w}^*)$ and each α_k, we introduce the deviation variable:

$$\varepsilon_k = orness(\mathbf{w}^*) - \alpha_k, (k = 1,\cdots,l).$$

We hope that each deviation variable $\varepsilon_k \ \ (k = 1,\cdots,l)$ tends to be zero as much as possible, and meanwhile, following Wang et al. [30], the aggregation operator weights should be equally important and all the arguments can be equally aggregated. Taking the orness constraint into consideration, the model should be expressed as making all the weights as close to each other as possible.

Based on the above analysis, we can construct a new nonlinear optimization model to determine the operator weights.

$$\begin{aligned} &Minimize\ \eta\sum_{k=1}^{l}\theta_k{\varepsilon_k}^2 + (1-\eta)\sum_{i=1}^{n-1}\left(\frac{w_i}{w_{i+1}}+\frac{w_{i+1}}{w_i}-2\right) \\ &s.t.\ \left\{\left[\sum_{j=1}^{n} w_j\left(\left(\frac{n-j}{n-1}\right)^{\gamma}-1\right)^{\lambda}\right]^{\frac{1}{\lambda}}+1\right\}^{\frac{1}{\gamma}} - \alpha_k = \varepsilon_k, \quad 0 \le \alpha_k \le 1, \\ &\sum_{j=1}^{n} w_j = 1, \quad w_j \ge 0, \quad j = 1,\cdots,n, \quad k = 1,\cdots,l, \end{aligned} \tag{5}$$

where η stands for the relative importance degree of total deviation $\sum_{k=1}^{l}\theta_k{\varepsilon_k}^2$ such that $0 < \eta < 1$, and θ_k denotes the relative importance weight of the kth DM $(k = 1,\cdots,l)$.The model (5) is nonlinear and the optimal weight vector $\mathbf{w}^* = \left(w_1^*, w_2^*, \cdots, w_n^*\right)$ can be obtained by using Matlab or LINGO software package.

If any individual weight vector is equal to the optimal weight vector, then ε_k will be zero. In this case, the model (5) will degenerate to the model (4) with the orness measures of the GOWPUA operator. So, the nonlinear model (5) is in fact an extension of the model (4). The main advantage of nonlinear model (5) is that it can not only minimize the differences between the degrees of orness provided by each DM and the degree of orness corresponding to an optimal weight vector, but also produce as equally important weights as possible. In addition, the model (5) for determining the GOWPUA operator weights does not follow a regular distribution, which is also the advantage of the model.

IV. A METHOD FOR MAGDM BASED ON THE GOWPUA OPERATOR

For a MAGDM problem, let $A=\{A_1,A_2,\cdots,A_m\}$ be a set of alternatives, $G=\{G_1,G_2,\cdots,G_n\}$ a set of attributes, $\mathbf{w}=(w_1,w_2,\cdots,w_l)$ a weight vector of attributes such that $\sum_{j=1}^{n} w_j=1$ and $w_j\geq 0$, $D=\{d_1,d_2,\cdots,d_l\}$ a finite set of DMs and $\boldsymbol{\omega}=(\omega_1,\omega_2,\cdots,\omega_n)$ a weight vector of DMs satisfying $\sum_{j=1}^{n}\omega_j=1$ and $\omega_j\geq 0$. Assume that α_k is the degree of orness provided by the kth DM $(k=1,\cdots,l)$, and θ_k is the relative importance weight of the kth DM $(k=1,\cdots,l)$. In addition, we suppose that each DM provides his/her own decision matrix $R^k=\left(r_{ij}^k\right)_{m\times n}(k=1,\cdots,l)$, where r_{ij}^k is given by the DM d_k for the alternative $A_i\in A$ w.r.t. the attributes $G_j\in G$.

Since different attributes have different measurement scales in MAGDM problem, it is necessary for the standardization of attributes to avoid the variance among different attributes. In this paper, we consider two attributes, i.e., profit type and cost type. Let I_1 be a set of benefit attributes and I_2 a set of cost attributes. Then the decision matrix R^k can be transformed into a corresponding decision matrix $X^k=\left(x_{ij}^k\right)_{m\times n}(k=1,\cdots,l)$ via the following formulas:

$$x_{ij}^k=\frac{r_{ij}^k}{\max_i r_{ij}^k},\ j\in I_1,\quad x_{ij}^k=\frac{\min_i r_{ij}^k}{r_{ij}^k},\ j\in I_2. \tag{6}$$

Based on the above explanation, an approach for MAGDM problem is developed based on the GOWPUA operator and the concrete steps are shown as follows.

Step 1. Standardize the decision matrixes. Based on the formulas (6), the decision matrixes $R^k=\left(r_{ij}^k\right)_{m\times n}(k=1,2,\cdots,l)$ can be transformed into standardization matrixes $X^k=\left(x_{ij}^k\right)_{m\times n}$ $(k=1,2,\cdots,l)$.

Step 2. Calculate the weight vector of attributes. By solving the model (5), the GOWPUA weight vector $\mathbf{w}=(w_1,w_2,\cdots,w_l)$ can be derived.

Step 3. Aggregate the decision matrixes into a collective decision matrix. According to Eq. (3), i.e., $x_{ij}=GOWPUA(x_{ij}^1,x_{ij}^2,\cdots,x_{ij}^l)$, we can aggregate all the decision matrixes $X^k(k=1,2,\cdots,l)$ into a collective decision matrix $X=(x_{ij})_{m\times n}$.

Step 4. Calculate the weight vector of the DMs. By solving the model (5), the weight vector of DMs $\boldsymbol{\omega}=(\omega_1,\omega_2,\cdots,\omega_n)$ can be obtained.

Step 5. Aggregate the collective overall preference value. According to Eq. (3), i.e., $x_i=GOWPUA(x_{i1},x_{i2},\cdots,x_{in})$, we aggregate the collective overall preference value x_i of the alternative A_i .

Step 6. Rank the collective overall preference value $x_i(i=1,2,\cdots,m)$ in descending order.

Step 7. Select the best alternative. Rank all the alternatives $A_i(i=1,2,\cdots,m)$ in descending order and consequently select the best one in the light of the aggregated value $x_i(i=1,2,\cdots,m)$.

V. CONCLUSION

In this paper, we developed the generalized ordered weighted power utility averaging (GOWPUA) operator, which is an extension of the GOWLA operator. The main character of the GOWPUA operator is that it introduced the risk attitude of DMs in the aggregation process. We investigated some properties of the GOWPUA operator and proved that it was commutative, monotonic, bounded and idempotent. In addition, we discussed the families of the GOWPUA operator and found that they included a wide range of aggregation operators such as the OWLGA operator, OWGA operator, OWA operator, OWHA operator, etc. To determine the GOWPUA operator weights, we addressed an orness measure of the GOWPUA operator and analyzed its properties. We developed a new nonlinear optimization model to determine the optimal weight vector of the GOWPUA operator. The main character of the model is that it considered different DMs may have different degrees of orness, and can aggregate all the individual sets of weights into an overall set of weights. Furthermore, a new approach for MAGDM was given based on the GOWPUA operator. This approach is also applicable to different group decision-making problems effectively such as human resource management and financial management, etc.

In further research, it would be very interesting to extend our analysis to the case of more sophisticated situation such as introducing the behavior theory of DMs in the GOWPUA operator. Nevertheless, we leave that point to future research, since our methodology cannot be applied to that extended framework, which will result in more sophisticated calculation and which we cannot tackle here.

ACKNOWLEDGEMENTS

The authors are very grateful to the editors and anonymous referees for their insightful and constructive comments and suggestions which have helped to improve the paper. This research is supported by the Natural Science Foundation of China under Grant No. 71271083, State Scholarship Fund and the Fundamental Research Funds for the Central Universities under Nos. 12ZX08, 2015XS33.

REFERENCES

[1] Vahdani B, Mousavi SM, Tavakkoli-Moghaddam R. A new design of the elimination and choice translating reality method for multi-criteria group decision-making in an intuitionistic fuzzy environment. Appl Math Model 2013; 37: 1781–1799.

[2] Bezerra FM, Melo P, Costa JP. Visual and Interactive Comparative Analysis of Individual Opinions: A Group Decision Support Tool. Group Decis Negot 2014; 23: 101-125.

[3] Verma R, Sharma BD. Fuzzy Generalized Prioritized Weighted Average Operator and its Application to Multiple Attribute Decision Making. Int J Intell Syst 2014; 29: 26-49.

[4] Fullér R, Majlender P. An analytic approach for obtaining maximal entropy OWA operator weights. Fuzzy Sets Syst 2003; 124: 53–57.

[5] Fullér R, Majlender P. On obtaining minimal variability OWA operator weights. Fuzzy Sets Syst 2003; 136: 203–215.

[6] Merigó JM. New extensions to the OWA operator and its application in business decision-making. PhD thesis, Department of Business Administration, University of Barcelona; 2008.

[7] Merigó JM, Casanovas M. Decision-making with distance measures and induced aggregation operators. Comput Ind Eng 2010; 60: 66–76.

[8] Merigó JM, Casanovas M. Fuzzy generalized hybrid aggregation operators and its application in decision-making. Int J Fuzzy Syst 2010; 12: 15–24.

[9] Merigó JM, Gil-Lafuente AM. New decision-making techniques and their application in the selection of financial products. Inform Sci 2010; 180: 2085–2094.

[10] Merigó JM, Gil-Lafuente AM. Fuzzy induced generalized aggregation operators and its application in multi-person decision-making. Expert Syst Appl 2011; 38: 9761–9772.

[11] Dujmović J, Tré GD. Multicriteria Methods and Logic Aggregation in Suitability Maps. Int J Intell Syst 2011; 26: 971-1001.

[12] Agarwal M, Hanmandlu M, Biswas K K. A Probabilistic and Decision Attitude Aggregation Operator for Intuitionistic Fuzzy Environment. Int J Intell Sys 2013; 28: 806–839.

[13] Stengel DN. Aggregating Incomplete Individual Ratings in Group Resource Allocation Decisions. Group Decis Negot 2013; 22: 235-258.

[14] Yager RR. On ordered weighted averaging aggregation operators in multicriteria decision-making. IEEE Trans Syst Man Cybern 1988; 18: 183–190.

[15] Yager RR. Generalized OWA aggregation operators. Fuzzy Optim Decis Ma 2004; 3: 93–107.

[16] Yager RR. Generalized Bonferroni mean operators in multi-criteria aggregation. Fuzzy Sets Syst 2010; 161: 2227–2242.

[17] Yager RR, Filev DP. Parameterized “andlike” and “orlike” OWA operators. Int J Gen Syst 1994; 22: 297–316.

[18] Llamazares B. An Analysis of Some Functions That Generalizes Weighted Means and OWA Operators. Int J Intell Syst 2013; 28: 380-393.

[19] Chiclana F, Herrera F, Herrera-Viedma E. The ordered weighted geometric operator: properties and applications. In: Proc 8th Int Conf on Information Processing and Management of Uncertainty in Knowledge-Based Systems, Madrid, Spain, 2000. pp 985–991.

[20] Zhou LG, Chen HY. Generalized Ordered Weighted Logarithm Aggregation Operators and Their Applications to Group Decision-making. Int J Intell Syst 2010; 25: 683–707.

[21] Yager RR, Filev DP. Induced ordered weighted averaging operators. IEEE Trans Syst Man Cybern 1999; 29: 141–150.

[22] O'Hagan M. Aggregating template or rule antecedents in real-time expert Systems with fuzzy set logic. In: 22nd Annual IEEE Asilomar Conf on Signals, Systems and Computers, Pacific Grove, CA, USA; 1988, pp. 681–689.

[23] Wang YM, Parkan C. A minimax disparity approach for obtaining OWA operator weights. Inform Sci 2005; 175: 20–29.

[24] Majlender P. OWA operators with maximal Rényi entropy. Fuzzy Sets Syst 2005; 155: 340-360.

[25] Amin GR, Emrouznejad A. An extended minimax disparity to determining the OWA operators weights. Comput Indus Eng 2006; 50: 312-316.

[26] Emrouznejad A, Amin GR. Improving minimax disparity model to determine the OWA operator weights. Inform Sci 2010; 180: 147-1485.

[27] Pratt J. Risk aversion in the small and in the large. Econometrica 1964; 32: 122-136.

[28] Arrow K. Aspects of the Theory of Risk-Bearing. Yrjo Hahnsson Foundation, Helsinki; 1965.

[29] Wang YM, Parkan C. A preemptive goal programming method for aggregating OWA operator weights in group decision-making. Inform Sci 2007; 177: 1867–1877.

[30] Wang YM, Luo Y, Liu XW. Two new models for determining OWA operator weights. Comput Ind Eng 2007; 52: 203-209.