# Theory of Dipole-Exchange Spin Waves in a Ferromagnetic Nanotube. Consideration of Volume and Surface Modes

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***Abstract*—** *The paper extends study of dipole-exchange spin waves in a ferromagnetic nanotube with a circular cross-section started by the author in the previous paper. The proposed model considers the magnetic dipole-dipole interaction, the exchange interaction, the anisotropy effects, the damping effects, the general boundary conditions and the existence of both volume and surface modes for the considered spin waves. Therefore, a new method of obtaining the values' spectrum of the orthogonal (to the translation axis) wavenumbers for the investigated spin waves is proposed in addition to the previously obtained dispersion law. The method is based on the application of general boundary conditions for the magnetic field on a superposition of the above-mentioned modes. The obtained spectrum is shown to be a quasi-one-dimensional one – similar to that in a thin ferromagnetic field – for typical ferromagnetic nanotubes. Exploitation of the above-mentioned method essentially extends the area of application of the obtained results compared to the previous paper.*

***Keywords*— Spin wave, Dipole-exchange theory, Ferromagnetic nanotube, Volume mode, Surface mode.**

## I. INTRODUCTION

At the present time, a variety of actual and prospective technologies are based on the applications of spin waves in nanosystems. In particular, such waves are promising for application in information technologies – for creating new data storage [1], transfer [1,2] and processing [3] devices. One of the key problems for developing such technologies is theoretical modeling of spin-wave processes in these nanosystems. Such modeling is required not only for direct applications of spin waves, but also for synthesizing materials with preset magnetic properties because these properties are often influenced by spin-wave processes. This modeling, in turn, requires deeper understanding of the corresponding processes in magnetic nanosystems. In the proposed paper, one of the problems of the above-described type is solved.

It has been shown by numerous studies that the properties of nanosystems – in particular, spin-wave properties – depend essentially on their size and shape. Unfortunately, a general theory of spin waves in magnetic nanosystems has not been created at the moment. Therefore, spin waves in nanosystems of different geometries are studied separately. Among the variety of magnetic nanosystems of different configurations, a special class is represented by shell-type ferromagnetic nanosystems (nanoshells, nanotubes and others). These nanosystems exhibit unique – not inherent to traditional continuous nanosystems – magnetic properties that are prospective for numerous technical applications. For instance, magnetic properties of such nanosystems can be regulated more flexibly than properties of corresponding continuous nanosystems. However, such nanosystems remain poorly researched at the moment. In particular, study of spin waves in synthesized recently magnetic nanotubes [4] represents an actual topic of research.

The paper continues the study of dipole-exchange spin waves in a ferromagnetic nanotube with a circular cross-section started by the author in the papers [5,6]. In the study, the magnetic dipole-dipole interaction, the exchange interaction, the anisotropy effects and the damping effects are considered. In the previous papers of the author [5,6], a dispersion relation for the above-described spin waves has been obtained. However, for complete description of the considered waves, this relation should be complemented by either a relation between the wavenumber components or values' spectrum of the orthogonal (to the tube axis) wavenumber component. For the most nanosystems, that represents more challenging task than just finding the dispersion relation. Moreover, for thin films and nanotubes one should consider existence of both volume and surface modes that are, generally speaking, hybridized in the considered cases. In the papers [5,6], only a volume spin wave mode has been considered and the above-mentioned spectrum has been obtained only for a very specific particular case (the material outside the nanotube has been assumed to be a high-conductivity metal) thus essentially limiting the area of application of the entire obtained result. The proposed paper overcomes this limitation by considering both modes' types and applying a different – essentially more general – method of obtaining the above-mentioned spectrum. As a result, the obtained values' spectrum of the orthogonal wavenumber component has an essentially wider range of applications. The obtained spectrum of wavenumbers is shown to have a quasi-one-dimensional form for the considered (thin) nanotubes.

## II. PROBLEM STATEMENT. MODEL DESCRIPTION

Let us consider a ferromagnetic nanotube – with a circular cross-section – composed of a uniaxial ferromagnet of the "easy axis" type. Let us denote the ferromagnet parameters as follows: the exchange constant $\alpha$, the uniaxial anisotropy parameter $\beta$, the gyromagnetic ratio $\gamma$, the ground state magnetization $\vec{M}_0$ (is considered constant inside the tube), the dissipation parameter $\alpha_G$ (the Hilbert term is used for consideration of the dissipation), the external magnetic field $\vec{H}_0^{(e)}$ (is also considered constant). We assume that both the easy magnetization axis (with the direction unit vector $\vec{n}$) and the external magnetic field inside the tube are directed along the tube translation axis – the axis Oz. Therefore, the ground state magnetization is also directed along Oz. Let us denote the internal tube radius *a* and the external radius *b* (see Fig.1.).

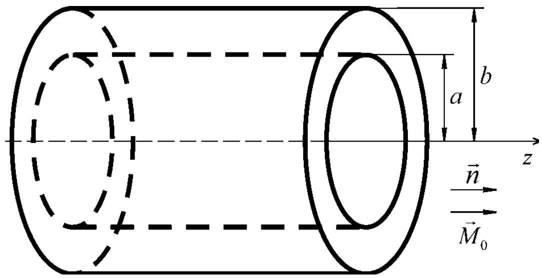


**FIGURE 1. The nanotube that is studied in the paper.**

Let us consider a spin wave propagating in the above-described nanotube. The magnetization $\vec{m}$ and the magnetic field $\vec{h}$ of the wave are assumed to be small perturbations of the overall magnetization $\vec{M}$ and the magnetic field inside the ferromagnet $\vec{H}^{(i)}$, correspondingly (linear wave). Thus, the relations $|\vec{m}| << |\vec{M}_0|$, $|\vec{h}| << |\vec{H}_0^{(i)}|$ fulfill, where $\vec{H}_0^{(i)}$ is the ground state internal magnetic field (so that $\vec{M} = \vec{M}_0 + \vec{m}$, $\vec{H}^{(i)} = \vec{H}_0^{(i)} + \vec{h}$). Let us consider the fact that for such spin waves, both volume and surface modes exist. The latter can make an essential contribution into spin-wave properties of the considered thin tube. Let us find the dispersion relation and the values' spectrum of the orthogonal (to the Oz axis) wavenumber for such linear spin waves.

For the investigated spin excitation, let us use the magnetostatic approximation, assuming that the magnetic potential Φ exists and, therefore, $\vec{h} = -\nabla\Phi$. After introducing amplitudes $\vec{m}_0$, $\vec{h}_0$ for the magnetization and the magnetic field perturbations, correspondingly (so that $\vec{m}(\vec{r},t) = \vec{m}_0(\vec{r})\exp(i\omega t)$, $\vec{h}(\vec{r},t) = \vec{h}_0(\vec{r})\exp(i\omega t)$, where $\omega$ is the wave frequency), we can write down the following relations for the magnetic potential: $\vec{h}_0 = -\nabla\Phi_0$, $\Phi = \Phi_0 \exp(i\omega t)$. The outside material is considered non-magnetic so the relations $\vec{m}_0 = 0$, $\Delta\Phi_0=0$ fulfill outside the investigated ferromagnet. After combining the linearized Landau-Lishitz equation with the Maxwell equation $div\vec{H}^{(i)} = -4\pi div\vec{M}$ the following starting system of equations can be written (see [5,6]):

$$\begin{cases} i\omega\vec{m}_0 = \gamma M_0 \vec{e}_z \times \left( \vec{h}_0 + \alpha \sum_i \frac{\partial^2 \vec{m}_0}{\partial x_i^2} - \left( \beta + \frac{H_0^{(i)}}{M_0} - i\frac{\alpha_G \omega}{\gamma M_0} \right) \vec{m}_0 \right), \\ \Delta\Phi_0 - 4\pi div\vec{m}_0 = 0 \end{cases} \quad (1)$$

here the cylindrical coordinates ($\rho,\theta,z$) are used. After considering the fact that for the considered nanosystem symmetry $\vec{H}_0^{(i)} = \vec{H}_0^{(e)} - 4\pi\hat{N}\vec{M}_0 = \vec{H}_0^{(e)}$ (here $\hat{N}$ is the demagnetizing coefficients tensor) and eliminating the magnetization amplitude in (1), the following equation for the magnetic potential of the investigated waves can be obtained (see [5,6]):

$$\left( \frac{\omega^2}{\gamma^2 M_0^2} - \left( \tilde{\beta} - i\frac{\alpha_G \omega}{\gamma M_0} - \alpha\Delta \right) \left( \tilde{\beta} - i\frac{\alpha_G \omega}{\gamma M_0} + 4\pi - \alpha\Delta \right) \right) \Delta\Phi_0 + 4\pi \left( \tilde{\beta} - i\frac{\alpha_G \omega}{\gamma M_0} - \alpha\Delta \right) \frac{\partial^2 \Phi_0}{\partial z^2} = 0, \quad (2)$$

here the value $\tilde{\beta} = \beta + H_0^{(e)}/M_0$.

The equation (2) should be complemented with the boundary conditions for the magnetic field. Analogously to the case of the ferromagnetic nanoshell (see the previous paper of the author [10]) let us assume that standard boundary conditions fulfill for the ground state magnetization and the magnetic field. Therefore, $b_{1n}=b_{2n}$, $h_{1\tau}=h_{2\tau}$ on the boundary of the considered ferromagnet (here medium 1 is the ferromagnet, medium 2 is the external medium, $n$ means normal and $\tau$ – tangential to the boundary vector component, $\vec{b}$ is the magnetic induction vector of the wave). For the vectors $\vec{h}$, $\vec{m}$ one can obtain $h_{1n}-h_{2n}=4\pi m_n$, $h_{1\tau}=h_{2\tau}$ (as the outside environment is non-magnetic). From these conditions and the condition of the potential continuity on the ferromagnet boundary, the following relations for the magnetic potential imply:

$$\Phi_0\big|_1 = \Phi_0\big|_2,\ \left(\nabla\Phi_0\right)_{1\tau} = \left(\nabla\Phi_0\right)_{2\tau},\ \left.\frac{\partial\Phi_0}{\partial n}\right|_1 - \left.\frac{\partial\Phi_0}{\partial n}\right|_2 = 4\pi m_{0n}. \tag{3}$$

As the conditions (3) contain not only the magnetic potential, but also a normal component of the magnetization of the wave, boundary conditions for the magnetic potential should be implied in addition to (3). However, as it will be shown further, for the considered nanosystem the sought orthogonal wavenumber spectrum does not depend on these conditions.

The system (2) together with the boundary conditions (3) will be used as starting relations during further investigation.

## III. SPECTRAL CHARACTERISTICS OF THE SPIN WAVES

Similarly to [5,6] let us seek a the potential inside the ferromagnet in the form

$$\Phi_0 = \left(A_1 J_n(k_\perp\rho) + A_2 N_n(k_\perp\rho)\right)\exp\left(i\left(n\theta + k_{||}z\right)\right) \tag{4}$$

that satisfies the Poisson equation $\Delta\Phi_0=k^2\Phi_0$. Here $A_1$, $A_2$ are constants, $k_\perp$ and $k_{||}$ are the orthogonal and longitudinal wavenumbers, correspondingly (they describe the wave propagation in the orthogonal to easy axis direction or along that direction, correspondingly), $n$ is the mode number and $J_n$, $N_n$ are the Bessel and Neumann functions of the order $n$, correspondingly.

$$\begin{aligned}&\alpha^2\left(k_{||}^2+k_\perp^2\right)^3 + 2\alpha\left(\beta+\frac{H_0^{(e)}}{M_0}-i\frac{\alpha_G\omega}{\gamma M_0}+2\pi\right)\left(k_{||}^2+k_\perp^2\right)^2 + \\ &+\left(\left(\beta+\frac{H_0^{(e)}}{M_0}-i\frac{\alpha_G\omega}{\gamma M_0}\right)\left(\beta+\frac{H_0^{(e)}}{M_0}-i\frac{\alpha_G\omega}{\gamma M_0}+4\pi\right)-\frac{\omega^2}{\gamma^2M_0^2}-4\pi\alpha k_{||}^2\right)\left(k_{||}^2+k_\perp^2\right)-4\pi\left(\beta+\frac{H_0^{(e)}}{M_0}-i\frac{\alpha_G\omega}{\gamma M_0}\right)k_{||}^2=0\end{aligned} \tag{5}$$

Substitution of the solution (4) into the equation (3) leads to the dispersion equation that is of the 6[th] order by $k_\perp$ for the given $k_{||}$, $\omega$. (The wavenumber $k_\perp$ has a discrete spectrum of values while $k_{||}$ is varying approximately continuously because the nanotube length is much greater than its thickness. Therefore, the component $k_{||}$ is chosen as the defining parameter.) Three roots (values of $k_\perp^2$) of this equation exists, one (with the real $k_\perp$ for $\alpha_G=0$) corresponds to the volume modes and other two (with complex values of $k_\perp$ that are complex-conjugate even for $\alpha_G=0$) correspond to surface modes that decay inside the tube from the internal ($\rho=a$, $\mathrm{Im}k_\perp>0$) and the external ($\rho=b$, $\mathrm{Im}k_\perp<0$) surfaces. Elementary spin excitation, therefore, should – in a general case – be a superposition of three excitations of the form (4), with three above-described values of $k_\perp$ (for the given $k_{||}$, $\omega$), denoted further as $k_\perp^{(1)}$ (volume mode) and $k_{\perp R}^{(2)} \pm ik_{\perp I}^{(2)}$ (surface modes). (Analogous effects for flat ferromagnetic films have been investigated, e.g., in [8]). In some cases this 3-modes elementary excitation can be separated into three individual modes that can exist independently. Therefore, the boundary conditions can be applied separately to the each mode.

Substitution of the potential (4) (regardless of whether it is single mode (4) or the above-described superposition) leads to the following dispersion law [6]:

$$\omega = \frac{|\gamma|M_0}{1+\alpha_G^2}\left(\sqrt{\left(1+\alpha_G^2\right)\left(\alpha^2k^4+2\alpha\tilde{\beta}k^2+\tilde{\beta}^2+4\pi\alpha\left(1+\frac{\tilde{\beta}}{\alpha k^2}\right)k_\perp^2\right)-\alpha_G^2\left(\frac{K}{k}\right)^4}-i\alpha_G\left(\frac{K}{k}\right)^2\right), \tag{6}$$

here $K^2=\alpha k^4+\tilde{\beta}k^2+2\pi k_\perp^2$, $k^2=k_{||}^2+k_\perp^2$ is the total wavenumber. One can notice that spin waves can only be excited for small values $\alpha_G\le 0.1$ and, therefore, in the dispersion law (4) the addend $\alpha_G^2$ can be neglected compared to 1. This allows rewriting the dispersion law (6) in the following simplified form:

$$\omega = |\gamma| M_0 \left( \sqrt{\alpha^2 k^4 + 2\alpha\tilde{\beta} k^2 + \tilde{\beta}^2 + 4\pi\alpha \left(1 + \frac{\tilde{\beta}}{\alpha k^2}\right) k_\perp^2 - 4\pi^2 \alpha_G^2 \frac{k_\perp^4}{k^4}} - i\alpha_G \left( \alpha k^2 + \tilde{\beta} + 2\pi \frac{k_\perp^2}{k^2} \right) \right). \quad (7)$$

Magnetic potential outside the tube $\Phi_0^e$ should satisfy the Laplace equation $\Delta\Phi_0=0$ and (together with its radial derivative) the continuity condition. Therefore, for a single mode (4) the outside potential has the following form:

$$\Phi_0^e = \begin{bmatrix} A_1^e I_n(k_{||}\rho)\exp(i(n\theta + k_{||}z)), & \rho \le a \\ A_2^e K_n(k_{||}\rho)\exp(i(n\theta + k_{||}z)), & \rho > b \end{bmatrix} \quad (8)$$

with the constants $A_1^e = (I_n(k_{||}a))^{-1}(A_1 J_n(k_\perp a) + A_2 N_n(k_\perp a)), A_2^e = (K_n(k_{||}b))^{-1}(A_1 J_n(k_\perp b) + A_2 N_n(k_\perp b))$. Here $I_n$, $K_n$ are the modified Bessel functions of the first and the second kind, correspondingly, of the order $n$.

In order to use the third boundary condition in (3), amplitude magnetization components $m_{0\rho}$, $m_{0\theta}$ should also be found (the component $m_{0z}$=0 because $\vec{M}_0$||Oz). These components should have the form $m_{0\rho} = f_1(\rho)\exp(i(n\theta + k_{||}z))$, $m_{0\theta} = f_2(\rho)\exp(i(n\theta + k_{||}z))$. Therefore, the following system of equations for $f_1$, $f_2$ can be obtained from (1):

$$\begin{cases} \alpha f_1'' + \left(\frac{3\alpha}{\rho} + \frac{\omega\rho}{\gamma M_0 n}\right) f_1' - \left(\alpha\left(\frac{n^2+1}{\rho^2} + k_{||}^2\right) + \left(\beta + \frac{H_0^{(e)}}{M_0} - i\frac{\alpha_G \omega}{\gamma M_0}\right)\right) f_1 - \frac{k^2 R}{4\pi}\left(\frac{2\alpha}{\rho} + \frac{\omega\rho}{\gamma M_0 n}\right) - R' = 0 \\ f_2 = -\frac{i\rho}{n}\left(\frac{k^2 R}{4\pi} - f_1'\right) \end{cases}, \quad (9)$$

here $R(\rho)$ is a radial part of the dependence (4). This function is a combination of expressions $A_1 J_n(k_\perp \rho) + A_2 N_n(k_\perp \rho)$ with three above-described different values of $k_\perp$ (one for the volume mode and two for the surface modes). The considered nanotube is thin ($b$-$a$<<$a$) and, therefore, inside the nanotube $k_\perp\rho \sim 2\pi\rho\,(b-a)^{-1}$>>1, $\rho\in[a,b]$ (for the complex values of $k_\perp$ this relation fulfils with $|k_\perp|$). Then, the asymptotics of the Bessel and Neumann functions can be used in (4): $J_n(k_\perp\rho) \approx \sqrt{2/(\pi k_\perp \rho)}\cos(k_\perp\rho - n\pi/2 - \pi/4)$, $N_n(k_\perp\rho) \approx \sqrt{2/(\pi k_\perp\rho)}\sin(k_\perp\rho - n\pi/2 - \pi/4)$. One can also note that for typical nanotubes their thickness is of the same order of magnitude as the exchange length (unities of nm for typical ferromagnets). Therefore, the factors that consider surface modes decay inside the nanotube can be considered approximately constant: $\exp(-k_{\perp I}^{(2)}\rho) \approx \exp(-k_{\perp I}^{(2)}a)$, $\exp(k_{\perp I}^{(2)}\rho) \approx \exp(k_{\perp I}^{(2)}b)$. Therefore, the function $R$ can be rewritten as

$$R \approx \frac{1}{\sqrt{\rho}}\left(A_1^{(1)}\cos\left(k_\perp^{(1)}\rho - \frac{n\pi}{2} - \frac{\pi}{4}\right) + A_2^{(1)}\sin\left(k_\perp^{(1)}\rho - \frac{n\pi}{2} - \frac{\pi}{4}\right) + A_1^{(2)}\cos\left(k_{\perp R}^{(2)}\rho - \frac{n\pi}{2} - \frac{\pi}{4}\right) + A_2^{(2)}\sin\left(k_{\perp R}^{(2)}\rho - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right), \quad (10)$$

where $A_1^{(1)}$, $A_2^{(1)}$, $A_1^{(2)}$, $A_2^{(2)}$ are constants. After substituting the 3-modes combination (10) for the magnetic potential into the system of equations (9) and then – after taking corresponding derivatives – replacing $\rho$ with its mean value $\rho_0=(a+b)/2$ in all power factors, one can obtain $f_1$ in the form

$$f_1 = \frac{1}{\sqrt{\rho}}\left(E_{11}\cos\left(k_\perp^{(1)}\rho - \frac{n\pi}{2} - \frac{\pi}{4}\right) + E_{12}\sin\left(k_\perp^{(1)}\rho - \frac{n\pi}{2} - \frac{\pi}{4}\right) + E_{21}\cos\left(k_{\perp R}^{(2)}\rho - \frac{n\pi}{2} - \frac{\pi}{4}\right) + E_{22}\sin\left(k_{\perp R}^{(2)}\rho - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right). \quad (11)$$

Here $E_{11}$, $E_{12}$, $E_{21}$, $E_{22}$ are constants. After substituting into the third boundary condition of system (3) the magnetic potential with the radial part in the form (10) (with replacement $\rho\to\rho_0$ in all power factors after taking radial derivatives) and the magnetization component $m_{0\rho}$ with the function $f_1$ in the form (11), one can notice that in the resulting system of equations, the values $(k_{||}/k_\perp)(I_n'(k_{||}a)/I_n(k_{||}a))$, $(k_{||}/k_\perp)(K_n'(k_{||}b)/K_n(k_{||}b))$ for both $k_\perp = k_\perp^{(1)}$, $k_\perp = k_{\perp R}^{(2)}$ can be neglected compared to unity when $k_{||}$<<$k_\perp$. (Really, the quantities $I_n'(k_{||}a)/I_n(k_{||}a)$, $K_n'(k_{||}b)/K_n(k_{||}b)$ are of the order of magnitude of $(k_{||}a)^{-1}$, $(k_{||}b)^{-1}$ correspondingly when $k_{||}a$<<1 and of the order of magnitude of unity or less on the remaining part of $k_{||}$ range. The quantity $k_\perp$ – when it is nonzero so the orthogonal spin excitations are present – is of the same order of magnitude as $(b-a)^{-1}$ and the nanotube is thin so that $b$-$a$<<$a$. Therefore, the quantities $(k_{||}/k_\perp)(I_n'(k_{||}a)/I_n(k_{||}a))$,

$\left(k_{\|}/k_{\perp}\right)\left(K_n'\left(k_{\|}b\right)/K_n\left(k_{\|}b\right)\right)$ really are small compared to 1 regardless of the value of $k_{\|}a$ as long as the relation $k_{\|}<<k_{\perp}$ fulfils.) The quantity $k_{\|}$ is of the same order of magnitude as the reverse length of the nanotube which is much less than $(b-a)^{-1}$, so the relation $k_{\|}<<k_{\perp}$ fulfils – and, therefore, the above-mentioned quantities can be neglected – on the most part of both quantities' ranges. As a result, the above-mentioned boundary condition can be rewritten into the form $G_1\cos\left(k_{\perp}^{(1)}a+\alpha_1\right)+G_2\cos\left(k_{\perp R}^{(2)}a+\alpha_2\right)=0$, $G_1\cos\left(k_{\perp}^{(1)}b+\alpha_1\right)+G_2\cos\left(k_{\perp R}^{(2)}b+\alpha_2\right)=0$. Here $G_1$, $G_2$, $\alpha_1$, $\alpha_2$ are constants, combinations of $A_1^{(1)}$, $A_2^{(1)}$, $A_1^{(2)}$, $A_2^{(2)}$, $E_{11}$, $E_{12}$, $E_{21}$, $E_{22}$, $k_{\perp}^{(1)}$ and $k_{\perp R}^{(2)}$. Therefore, regardless of the exact form of the boundary conditions for the magnetization, the orthogonal wavenumber has the quasi-one-dimensional form: $k_{\perp}(b-a)=0$ for both values of $k_{\perp}$ and, correspondingly,

$$k_{\perp}^{(1)}=\pi s_1/(b-a),\ k_{\perp R}^{(2)}=\pi s_2/(b-a). \qquad (12)$$

Here $s_1$, $s_2$ are non-negative integers (orthogonal modes' numbers). Therefore, the dispersion law for the investigated spin waves is given by the relation (7) with the orthogonal wavenumbers given by (12). For the volume mode these two relations completely describe the spectral characteristics of the wave.

## IV. DISCUSSION

Let us analyze the obtained dispersion relation for the volume spin wave mode.

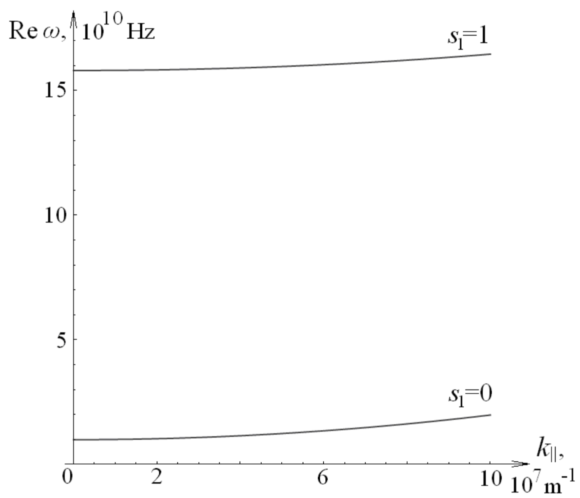


**FIGURE 2. The dependence of Re$\omega$ on $k_{\perp}$ for $\beta$=1, $\alpha$=10$^{-12}$ cm$^{-2}$, $\gamma$=10$^{7}$ Hz/Gs, $M_0$=10$^{3}$ Gs. The first two radial modes ($s_1$=0, $s_1$=1) are shown.**

As it can be seen from (7), the dependence of the imaginary part of the frequency (that describes to the spin wave decay) on the longitudinal wavenumber $k_{\|}$ in the investigated wavenumbers' range $k_{\|}<<k_{\perp}$ is close to the square law: $\mathrm{Im}\omega\approx|\gamma|M_0\alpha_G(\alpha(k_{\|}^2+k_{\perp}^2)+\tilde{\beta}+2\pi)$. On the other hand, the dependence of the real part of the frequency on $k_{\|}$ is more complex but close to a constant if the orthogonal excitations are present ($k_{\perp}\neq0$). The mode with $s_1$=0, $k_{\perp}$=0 formally does not satisfy the relation $k_{\|}<<k_{\perp}$; however, such mode is known to exist and is the only possible mode when the nanotube thickness is less than the exchange length. The dependence of Re$\omega$ on $k_{\perp}$ for typical values on the nanotube parameters is depicted on the Fig. 2. The first two radial modes ($s_1$=0, $s_1$=1) are presented on the graph.

The longitudinal wavenumber is restricted, on the one hand, by the nanotube length – unities of micrometers for typical nanotubes – and, on the other hand, by the interatomic distance $d_0$ – several angstroms for typical materials. Therefore, the wavenumber lies in the interval $10^6$-$10^9$ m$^{-1}$. Substitution of the above-mentioned typical values of nanotube parameters ($\beta$=1, $\alpha$=10$^{-12}$ cm$^{-2}$, $\gamma$=10$^{7}$ Hz/Gs, $M_0$=10$^{3}$ Gs) into the obtained dispersion law shows that the real part of the spin wave frequency lies in the interval $10^{10}$-$10^{12}$ Hz. This, really, is a frequencies interval for typical observed spin waves.

## V. CONCLUSION

Thus, dipole-exchange spin waves in a ferromagnetic nanotube (easy-axis ferromagnet) have been studied in the paper. The magnetic dipole-dipole interaction, the exchange interaction, the anisotropy effects and the damping effects have been considered. It has been shown that – similarly to the known case of a thin ferromagnetic film – in the considered nanotube, both surface and volume spin wave modes can be observed as well as hybrid modes. The dispersion law for such waves – obtained in the previous paper of the author – has been complemented with the spectrum of orthogonal wavenumber values. Unlike the previous papers of the author [5,6], the above-mentioned spectrum have been obtained using general boundary conditions for the magnetic field. This essentially extends the area of application of the obtained results compared to the papers [5,6] in which the spectrum has been found for a very specific particular case. It has been shown that for both modes' types the orthogonal wavenumber (for the surface modes – its real part) values' spectrum is quasi-one-dimensional on the most part of orthogonal and longitudinal wavenumbers' ranges. Therefore, the obtained results can be used for any round ferromagnetic nanotube of the studied configuration as long as the general model used in the paper (thin nanotube, linear waves, constant absolute value of the magnetization vector etc.) can be applied – and the mentioned model is applicable for typical round ferromagnetic nanotubes synthesized nowadays.

A graphical representation of the resulting dispersion relation with account for the obtained orthogonal wavenumber values' spectrum has been given. Numerical estimations of the spin waves' frequency for typical nanoshell parameters have been performed. The estimations have shown that the resulting frequency, really, lies within the frequencies interval for typical observed spin waves.

The method proposed in the paper can be applied to nanotubes of more complex configurations – in particular, to synthesized recently ferromagnetic nanotubes with an elliptic cross-section – as well as for more complex configurations of shell-type nanosystems in general. For some of them boundary conditions can be applied separately on volume and surface modes.

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