Hopf Bifurcation Analysis for the Comprehensive National Strength Model with Time Delay

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***Abstract*—** *This paper mainly modifies and further develops the comprehensive national strength model. By modifying the basic comprehensive national strength model, it can more accurately illustrate the society phenomena with time delay. First, we research the dynamics of the modified with time delay. By employing the normal form theory and center manifold method, we obtain some testable results on these issues. The conclusion confirms that a Hopf bifurcation occurs due to the existence of stability switches when the delay varies. Finally, some numerical simulations are given to illustrate the effectiveness of our result.*

***Keywords*— *Hopf bifurcation, Stability, Comprehensive national strength model, Center manifold, Normal form*.**

I. THE ESTABLISHMENT OF THE MODEL

Ordinary differential equation is used to study the comprehensive national strength of a country's overall situation, which is gradually developed in recent years. This article brings in time delay in the comprehensive national strength model. Besides, the comprehensive national strength model models are detailed analyzed. $x = x(t)$ is hard power strength in t, which is a composite indicator the level of the material civilization (resource, economy, military, science and technology, etc.) of a country. As $x(t)$ bigger, the material civilization more prosperous. $y(t) > 0$ is soft power strength in t, which is the spiritual civilization of a country. $y(t) > 0$ stand for social evils (decision-making errors, education failure, official corruption people steal, etc.). At this time, soft power has obstacle to the social development. $y(t) > 0$ mean soft power is superior (wise decision-making, national quality, etc.), that has a promoting effect on social development. α, β, γ, m, M is constant. So, in the literature [1] comprehensive national strength shown by the following differential equation:

$$\begin{cases} \dot{x} = \alpha x(\dfrac{M-x}{M}) - \beta y, \\ \dot{y} = -\gamma y + \delta(m-x)x. \end{cases} \tag{1.1}$$

As is known to all, some change of the soft power can be reflected on the hard power after a certain period of time, the literature [2] has established the following delay model of the comprehensive national strength.

$$\begin{cases} \dot{x} = \alpha x(\dfrac{M-x}{M}) - \beta y(t-\tau), \\ \dot{y} = -\gamma y + \delta(m-x)x, \end{cases} \tag{1.2}$$

where is positive, and the other parameters are the same as of (1.1). While model of (1.2) had considered the time delay, in the real world, hard power strength not only rely on hard power strength of soft power in the past, but also rely on soft power. In order to make the model more accurate, we modify the system of (1.2) to the following form:

$$\begin{cases} \dot{x} = \alpha x(\dfrac{M-x}{M}) - \beta(ky(t-\tau) + dy), \\ \dot{y} = -\gamma y + \delta(m-x)x, \end{cases} \tag{1.3}$$

where $k+d=1$. The organization of this paper is as follows: regarding τ as bifurcation parameter, we study the stability of the equilibrium point of the system (1.3) and Hopf bifurcation of the equilibrium depending on τ. Then, based on the new normal form of the differential-algebraic system introduced by Chen et al. [3] and the normal form approach theory and center manifold theory introduced by Hassard et al. [4], we derive the formula for determining the properties of Hopf bifurcation of the system in the third section. Numerical simulations aimed at justifying the theoretical analysis will be reported in Section 4. Finally, this paper ends with a discussion.

II. STABILITY AND LOCAL HOPF BIFURCATION ANALYSIS

The stability and Hopf conclusions for the system of (1.1) can be obtained directly from the literature [5,6]. From system (1.3), we can see that there exists equilibriums.

$$E_0=(X_0,Y_0)=(\frac{M(\alpha\gamma-m\beta\delta)}{\alpha\gamma-M\beta\delta},\frac{M\alpha\delta(m-M)(\alpha\gamma-m\beta\delta)}{(\alpha\gamma-M\beta\delta)^2}),E_1=(0,0).$$

According to the practical significance of the model, here we only discuss the problems of the hopf bifurcation and stability for the sole positive equilibrium point E_0.

Set $u_1(t)=x(t)-X_0$, $u_2(t)=y(t)-Y_0$. The system (1.3) becomes

$$\begin{cases}\dot{u}_1(t)=\alpha(1+p)u_1-\dfrac{\alpha}{M}u_1^2-\beta(ku_2(t-\tau)+du_2),\\ \dot{u}_2(t)=-\gamma u_2+\delta(m+Mp)u_1-\delta u_1^2.\end{cases} \tag{2.1}$$

Where $p=\dfrac{2(\alpha\gamma-m\beta\delta)}{-\alpha\gamma+M\beta\delta}$. And, the linearization of system (2.1) at E_0 is

$$\begin{cases}\dot{u}_1(t)=\alpha(1+p)u_1-\beta(ku_2(t-\tau)+du_2),\\ \dot{u}_2(t)=-\gamma u_2+\delta(m+Mp)u_1.\end{cases} \tag{2.2}$$

The characteristic equation of system (2.2) is as follows.

$$\lambda^2-c\lambda-ge^{-\lambda\tau}+f=0, \tag{2.3}$$

where $c=\alpha(1+p)+\delta(m+Mp),\ f=\alpha\delta(1+p)(m+Mp)-d\gamma\beta,\ g=\gamma\beta k.$ To study the stability of the equilibrium point E_0 and bifurcation system, we only need to discuss the distribution of the roots for characteristic Eq. (2.3). If the all roots of equation (2.3) have negative real part, the equilibrium point E_0 is steady. If the equation has a root that contains positive real part, the equilibrium point E_0 is not stable. In order to study the distribution of the roots for Eq. (2.3). Considering $\tau=0$ in first, characteristic equation (2.3) is

$$\lambda^2-c\lambda+q=0. \tag{2.4}$$

Where $q=\alpha\delta(1+p)(m+Mp)-\gamma\beta$.

$$(H_1)c<0,q>0.$$

Obviously, all roots of equation of (2.3) has negative real part if (H_1) is satisfied. So the system in equilibrium E_0 is locally asymptotically stable for $\tau = 0$. Now, we investigate the local stability around the positive equilibrium point for the system (1.3) and the existence of Hopf bifurcations occurring at the equilibrium point E_0 when $\tau_0 > 0$.

Lemma 2.1 For the system (1.3), Eq.(2.2) has a part of purely imaginary roots for $\tau_0 > 0$ when (H_1) is satisfied.

Proof. If $\lambda = i\omega$ is a solution of the characteristic equation (2.2), when and only when ω meet

$$-\omega^2 - c\omega i - g(\cos\omega\tau - i\sin\omega\tau) + f = 0.$$

The separation of the real and imaginary parts yields.

$$\begin{cases} -\omega^2 + f = g\cos\omega\tau, \\ c\omega = g\sin\omega\tau, \end{cases} \tag{2.5}$$

which lead to

$$v^2 + (c^2 - 2f)v + f^2 - g^2 = 0, \tag{2.6}$$

where $v = \omega^2$. We assume that the coefficients satisfies the following conditions.

$$(H_{21})c^2 - 2f < 0, 4(f^2 - g^2) \le (c^2 - 2f)^2, (H_{22})c^2 - 2f > 0, 4(f^2 - g^2) < 0.$$

If the condition (H_{21}) or (H_{22}) is satisfied, then Eq. (2.6) has positive roots. Therefore, Eq. (2.2) have purely imaginary roots. From (2.6) we obtain

$$\omega_0 = \sqrt{\frac{2f - c^2 + \sqrt{(c^2 - 2f)^2 - 4(f^2 - g^2)}}{2}},$$

$$\tau_j = \frac{1}{\omega_0}(\arccos(\frac{f - (\omega_0)^2}{g}) + 2j\pi), j = 0,1,2,\cdots.$$

As a result, when $\tau = \tau_j$, the characteristic equation have a pair of purely imaginary root. The proof is completed.

Lemma 2.2 The transversality conditions $\mathrm{Re}(\frac{d\lambda}{d\tau_0})^{-1} > 0$ or (H_{22}) is satisfied.

Proof. By differentiating both sides of Eq. (2.3) with regard to τ and solving $\lambda'(\tau)$. We have

$$\frac{d\lambda}{d\tau} = \frac{-g\lambda e^{-\lambda\tau}}{2\lambda - c + g\tau e^{-\lambda\tau}},$$

then

$$\mathrm{Re}(\frac{d\lambda}{d\tau})^{-1}\Big|_{\substack{\lambda=i\omega_0 \\ \tau=\tau_j}} = \mathrm{Re}(\frac{2\lambda - c + g\tau e^{-\lambda\tau}}{-g\lambda e^{-\lambda\tau}})\Big|_{\substack{\lambda=i\omega_0 \\ \tau=\tau_j}}$$

$$=\frac{-2\omega_0\cos\omega_0\tau_j+c\sin\omega_0\tau_j}{g\omega_0}$$

$$=\frac{c^2-2f+2\omega^2}{g^2}.$$

We can get $c^2-2f+2\omega^2>0$ by condition (H_{21}) or (H_{22}), thus $\mathrm{Re}(\frac{d\lambda}{d\tau_0})^{-1}>0$. The proof is completed.

Lemma 2.3 For Eq. (2.3), if $\tau\in[0,\tau_0)$, all of his roots have negative real part. Positive equilibrium is asymptotically stable, and the positive equilibrium produces Hopf bifurcation in $\tau=\tau_0$.

III. THE DIRECTION OF THE HOPF BIFURCATION AND THE STABILITY OF PERIODIC SOLUTIONS

In §2, we obtain the conditions of Hopf bifurcation. In this section, we discuss the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions based on the normal form introduced by Chen et al. [3] and the center manifold theory introduced by Hassard et al. [4].

In the following part, we assume that the system (1.3) undergoes Hopf bifurcation at the equilibrium E_0 for $\tau=\tau_j$, and we let $i\omega_0$ is the corresponding purely imaginary root of the characteristic equation at the equilibrium E_0. Set $\tau=\tau_0+\mu,\mu\in R$, clearly, $\mu=0$ is the Hopf bifurcation value of system (3.2). Set $t=s\tau$, $\bar{u}_i(t)=u_i(t\tau),i=1,2$, for convenience, we continue to use $u_i(t)$ said $\bar{u}_i(t)$.Then the system (1.3) is equivalent to the following Functional Differential Equation (FDE) system in $C\in C([-1,0],R^2)$,

$$\dot{u}(t)=L_\mu+F(\mu,u_t). \tag{3.1}$$

where $u(t)=(u_1(t),u_2(t))^T$, $L_\mu:C\to R,F:R\times C\to R$ are given, respectively, by

$$L_\mu(\phi)=(\tau_0+\mu)B_1\phi(0)+(\tau_0+\mu)B_2\phi(-1),\ F(\mu,\phi)=(\tau_0+\mu)\begin{pmatrix}-\frac{\alpha}{M}\phi_1^2(0)\\ -\delta\phi_2^2(0)\end{pmatrix},$$

where L_μ为 $\phi=(\phi_1,\phi_2)^T\in C([-1,0],R^2)$.

$$B_1=\begin{pmatrix}\alpha(1+p) & -\beta d\\ -\gamma & \delta(m+Mp)\end{pmatrix},B_2=\begin{pmatrix}0 & -\beta k\\ 0 & 0\end{pmatrix}.$$

By the Riesz representation theorem, there exists a matrix function whose components are bounded variation function $\eta(\theta,\mu):[-1,0]\to R^{2\times2}$ such that

$$L_\mu\phi=\int_{-1}^0 d\eta(\theta,\mu)\phi(\theta),\phi\in C. \tag{3.2}$$

In fact, we take

$$\eta(\theta,\mu)=B_1\delta(\theta)+B_2\delta(\theta+1). \tag{3.3}$$

$\delta(\theta)$ is a Delta function. For $\phi\in C^{'}([-1,0],R^2)$, we define operator A and R as follows.

$$A(\mu)\phi(\theta)=\begin{cases}\dfrac{d(\phi(\theta))}{d\theta}, & \theta\in[-1,0),\\ \int_{-1}^{0}d(\eta(\theta,\mu)\phi(\theta)), & \theta=0.\end{cases} \tag{3.4}$$

$$R(\mu)\phi(\theta)=\begin{cases}0, & \theta\in[-1,0),\\ F(\mu,\phi), & \theta=0.\end{cases} \tag{3.5}$$

Then the system (2.1) can be written as the following form:

$$\dot{u}_t=A(\mu)u_t+R(\mu)u_t \tag{3.6}$$

Setting $\varphi\in C^{'}([0,1],(R^2)^*)$, the adjoint operator A^* of A is defined as

$$A*\psi(s)=\begin{cases}-\dfrac{d\psi(s)}{ds}, & s\in(0,1],\\ \int_{-1}^{0}d(\eta^T(t,0)\psi(-t)), s=0,\end{cases} \tag{3.7}$$

and a bilinear inner product is given by

$$<\varphi(s),\phi(\theta)>=\overline{\varphi}(0)\phi(0)-\int_{-1}^{0}\int_{\varepsilon=0}^{\theta}\overline{\varphi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi. \tag{3.8}$$

From the discussions in Section 2, we know that $i\tau_j\omega_0$ are eigenvalues of A. Thus, they are also eigenvalues of A^*. Next we calculate the eigenvector $q(\theta)$ of A belonging to $i\tau_j\omega_0$ and eigenvector $q^*(s)$ of A^* belonging to the eigenvalue $-i\tau_j\omega_0$. By the definition of A, we have $A(0)q(0)=\dfrac{dq(\theta)}{d\theta}$ and $q(\theta)=q(0)e^{i\tau_j\omega_0\theta}$. In addition,

$$\int_{-1}^{0}d\eta(\theta)q(\theta)=B_1q(0)+B_2q(-1)=A(0)q(0)=i\omega_0q(0), \tag{3.9}$$

Let $q(0)=(1,q_2)^T$, Eq.(3.9) becomes

$$\begin{pmatrix}\alpha(1+p) & -\beta(d+ke^{-i\omega_0\tau_j})\\ -\gamma & \delta(m+Mp)\end{pmatrix}\begin{pmatrix}1\\ q_2\end{pmatrix}=i\omega_0\begin{pmatrix}1\\ q_2\end{pmatrix},$$

where $q_2=\dfrac{\alpha(1+p)-i\omega_0}{\beta(d+ke^{-i\omega_0\tau_j})}$.

By the definition of A^*, we have $A^*(0)q^*(s)=-\dfrac{dq^*(s)}{ds}$ and $q^*(s)=q^*(0)e^{i\tau_0\omega_0 s}$. Then,

$$\int_{-1}^{0} q^*(-t)d\eta(t) = B_1^T q^*(0) + B_2^T q^*(-1) = A^* q^*(0) = -i\omega_0 q^*(0). \tag{3.10}$$

Let $q^*(0) = D(1, q_2^*)^T$, Eq.(3.10) becomes

$$\begin{pmatrix} \alpha(1+p) & -\gamma \\ -\beta(d+ke^{-i\omega_0\tau_j}) & \delta(m+Mp) \end{pmatrix} \begin{pmatrix} 1 \\ q_2^* \end{pmatrix} = -i\omega_0^* \begin{pmatrix} 1 \\ q_2^* \end{pmatrix},$$

where $q_2^* = \dfrac{\alpha(1+p)+i\omega_0}{\gamma}$.

$$< q^*(s), q(\theta) > = \overline{q}^*(0)q(0) - \int_{-1}^{0}\int_{\varepsilon=0}^{\theta} \overline{q}^*(\xi-\theta)d\eta(\theta)q(\xi)d\xi.$$

$$= \overline{D}[1 + q_2\overline{q_2}^* - \int_{-1}^{0}(1,\overline{q}_2^*)\theta e^{-i\tau_j\omega_0\theta}d\eta(\theta)\begin{pmatrix} 1 \\ q_2 \end{pmatrix}]$$

$$= \overline{D}[1 + q_2\overline{q_2}^* + \tau_j e^{-i\omega_0\tau_j}(1,\overline{q}_2^*)B_2\begin{pmatrix} 1 \\ q_2 \end{pmatrix}]$$

$$= \overline{D}[1 + q_2\overline{q}_2^* - \beta k\tau_j q_2 e^{-i\omega_0\tau_j}],$$

where $\langle q^*(s), q(\theta)\rangle = 1$, so $\overline{D} = [1 + q_2\overline{q}_2^* - \beta k\tau_j q_2 e^{-i\omega_0\tau_j}]^{-1}$.

Next, we study the specific parameters of the direction and size for bifurcated periodic solutions. Using the same notations as in Hassard et al. [4], we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Define

$$z(t) =< q*, u_t >, W(t,\theta) = u_t(\theta) - 2\,\mathrm{Re}\{z(t)q(\theta)\}. \tag{3.11}$$

On the center manifold C_0 ,we have

$$W(t,\theta) = W(z(t),\overline{z}(t),\theta) \tag{3.12}$$

Where $W(z,\overline{z},\theta) = W_{20}(\theta)\dfrac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\dfrac{\overline{z}^2}{2} + \cdots$.

In fact, z and $\overline{z}$ are local coordinates for center manifold C_0 in the direction of q and $\overline{q}$. Note that W is real if u_t is real. We consider only real solutions. For the solution of Eq.(3.6) in center manifold C_0, we have

$$z'(t) = \langle q^*(s), \dot{\mu}_t\rangle = \langle q*(\theta), (A(0)+R(0))\mu_t\rangle$$

$$= \langle A^*q^*(s), \mu_t\rangle + \overline{q}^*(s)R(0)\mu_t - \int_{-1}^{0}\int_{\varepsilon=0}^{\theta} \overline{q}^*(\xi-\theta)d\eta(\theta)A(0)R(0)\mu_t(\xi)d\xi$$

$$= \langle i\tau_j\omega_0 q^*(s), \mu_t\rangle + \overline{q}^*(0)F(0,\mu_t(\theta)) \tag{3.13}$$

Let

$$z^{'}(t)=i\omega_0 z+g(z,\bar{z}), \tag{3.14}$$

where

$$g(z,\bar{z})=g_{20}\frac{z^2}{2}+g_{11}z\bar{z}+g_{02}\frac{\bar{z}^2}{2}+\cdots, \tag{3.15}$$

from (3.13) and (3.15), we have

$$g(z,\bar{z})=\bar{q}^*(0)F_0(z,\bar{z})=\tau_j\bar{V}\begin{pmatrix}1\\ \bar{q}_2^*\end{pmatrix}\begin{pmatrix}F_1\\ F_2\end{pmatrix}=\tau_j\bar{V}(F_1+\bar{q}_2^*F_2). \tag{3.16}$$

By (3.11), we have

$$u_t(\theta)=(u_1(t+\theta),u_2(t+\theta))=W(t,\theta)+z(t)q(\theta)+\bar{z}(t)\bar{q}(\theta) \tag{3.17}$$

where

$$u_1(t+\theta)=W^{(1)}(t,\theta)+z(t)q^{(1)}(\theta)+\bar{z}(t)\bar{q}^{(1)}(\theta),$$

$$u_2(t+\theta)=W^{(2)}(t,\theta)+z(t)q^{(2)}(\theta)+\bar{z}(t)\bar{q}^{(2)}(\theta).$$

Then

$$g(z,\bar{z})=\tau_j\bar{D}[(\frac{-\alpha}{M}-\delta\bar{q}_2^*q_2^2)z^2+(\frac{-2\alpha}{M}-2\delta\bar{q}_2^*q_2q_2^2)z\bar{z}+(\frac{-\alpha}{M}-\delta\bar{q}_2^*\bar{q}_2^2)\bar{z}^2+$$
$$[\frac{-\alpha}{M}(2W_{11}^{(1)}(0)+\frac{1}{2}W_{20}^{(1)}(0)+\frac{1}{2}W_{20}^{(2)}(0))-\delta\bar{q}_2^*(2W_{11}^{(2)}(0)q_2+W_{20}^{(2)}(0)\bar{q}_2)]z^2\bar{z}+\cdots$$

Comparing the coefficients with (3.16), it follows that

$$g_{20}=2\tau_j\bar{D}(\frac{-\alpha}{M}-\delta\bar{q}_2^*q_2^2),$$

$$g_{11}=\tau_j\bar{D}(\frac{-2\alpha}{M}-2\delta\bar{q}_2^*q_2q_2^2),$$

$$g_{02}=2\tau_j\bar{D}(\frac{-\alpha}{M}-\delta\bar{q}_2^*\bar{q}_2^2),$$

$$g_{21}=2\tau_j\bar{D}[\frac{-\alpha}{M}(2W_{11}^{(1)}(0)+\frac{1}{2}W_{20}^{(1)}(0)+\frac{1}{2}W_{20}^{(2)}(0))-\delta\bar{q}_2^*(2W_{11}^{(2)}(0)q_2+W_{20}^{(2)}(0)\bar{q}_2)].$$

By Eq.(3.15), (3.16) and definition (3.11) of $W(t,\theta)$, we have

$$\dot{W} = \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q} = \begin{cases} A(0)W - 2\operatorname{Re}\bar{q}^*(0)F_0(z(t),\bar{z}(t))\bar{q}(\theta), & \theta \in [-1,0), \\ A(0)W - 2\operatorname{Re}\bar{q}^*(0)F_0(z(t),\bar{z}(t))q(\theta) + F_0, & \theta = 0. \end{cases}$$

$$\overset{\Delta}{=} AW + H(z,\bar{z},\theta) \tag{3.18}$$

Let $\dot{W} = A(0)W + H(z,\bar{z},\theta)$. Where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots \tag{3.19}$$

From the definition (3.11) of $W(t,\theta)$, we have

$$\dot{W} = W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}} \tag{3.20}$$

By (3.18), (3.19) and (3.20), we have

$$\begin{aligned} H_{20}(\theta) &= 2i\tau_0\omega W_{20}(\theta) - A(0)W_{20}(\theta), \\ H_{11}(\theta) &= -A(\theta)W_{11}(\theta). \end{aligned} \tag{3.21}$$

For $\theta \in [-1,0]$, by (3.14), (3.18) and (3.19), we get

$$H(z,\bar{z},0) = -2\operatorname{Re}[\bar{q}^*(0)F_0q(\theta)]$$

$$= -(g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{\bar{z}z^2}{2})q(\theta)$$

$$-(\bar{g}_{20}\frac{z^2}{2} + \bar{g}_{11}z\bar{z} + \bar{g}_{02}\frac{\bar{z}^2}{2} + \bar{g}_{21}\frac{\bar{z}z^2}{2})\bar{q}(\theta) \tag{3.22}$$

Comparing the coefficients with (3.19), we gives that

$$\begin{aligned} H_{20}(\theta) &= -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \\ H_{11}(\theta) &= -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \end{aligned} \tag{3.23}$$

It follows from (3.21) that

$$\begin{aligned} \dot{W}_{20}(\theta) &= 2i\omega\tau_0W_{20}(\theta) - A(0)g_{20}q(\theta) + \bar{g}_{20}\bar{q}(\theta), \\ \dot{W}_{11}(\theta) &= g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta). \end{aligned} \tag{3.24}$$

Solving for $W_{20}(\theta)$ and $W_{11}(\theta)$, we obtain

$$W_{20}(\theta)=\frac{ig_{20}}{\tau_0\omega}q(0)e^{i\tau_0\omega\theta}+\frac{i\overline{g}_{02}}{3\tau_0\omega}\overline{q}(0)e^{-i\tau_0\omega\theta}+E_1e^{2i\tau_0\omega\theta},$$
$$W_{11}(\theta)=-\frac{ig_{11}}{\tau_0\omega}q(0)e^{i\omega\tau_0\theta}+\frac{i\overline{g}_{11}}{\tau_0\omega}\overline{q}(0)e^{-i\omega\tau_0\theta}+E_2. \tag{3.25}$$

By (3.4) and (3.21), we get

$$\int_{-1}^{0}d\eta(\theta)W_{20}(\theta)=2i\tau_j\omega_0W_{20}(0)-H_{20}(0),$$
$$\int_{-1}^{0}d\eta(\theta)W_{11}(\theta)=-H_{11}(0) \tag{3.26}$$

From (3.17) and (3.18), we get

$$H_{20}(\theta)=-g_{20}q(0)-\overline{g}_{02}\overline{q}(0)+2\tau_j(E_{11},E_{12},E_{13})^T,$$
$$H_{11}(\theta)=-g_{11}q(0)-\overline{g}_{11}\overline{q}(0)+\tau_j(E_{21},E_{22},E_{23})^T. \tag{3.27}$$

According to Eq.(3.25), (3.26) and (3.27), we get

$$(2i\tau_j\omega_0I-\int_{-1}^{0}e^{2i\tau_j\omega_0}d\eta(\theta))E_1=2\tau_j(E_{11},E_{12},E_{13})^T, \tag{3.28}$$

Then

$$E_1=2\begin{pmatrix}2i\omega_0-\alpha(1+p) & \beta(d+ke^{-2i\omega_0\tau_j})\\ \gamma & 2i\omega_0-\delta(m+Mp)\end{pmatrix}^{-1}\begin{pmatrix}-\dfrac{\alpha}{M}\\ -\delta q_2^2\end{pmatrix},$$

$$E_2=-\begin{pmatrix}\alpha(1+p) & -\beta\\ -\gamma & \delta(m+Mp)\end{pmatrix}^{-1}\begin{pmatrix}-\dfrac{2\alpha}{2}\\ -2\delta q_2\overline{q}_2\end{pmatrix}.$$

According to the above Proposition and [7,11], we can compute the following parameters:

$$C_1(0)=\frac{i}{2\tau_j\omega_0}(g_{20}g_{11}-2|g_{11}|-\frac{1}{3}|g_{02}|^2)+\frac{g_{21}}{2},$$
$$\mu_2=-\frac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_j)\}},$$
$$\beta_2=2\,\mathrm{Re}\{C_1(0)\},$$
$$T_2=-\frac{\mathrm{Im}\{C_1(0)\}+\mu_2(\mathrm{Im}\{\lambda'(\tau_j)\})}{\omega_0\tau_j}.$$

IV. NUMERICAL SIMULATION

Numerical simulation shows from stable to unstable complex transformation process the system (1.3). We give a concrete example to show the dynamic behavior of comprehensive national strength model. We take $\alpha=0.1, M=10, \beta=0.3, k=0.8, d=0.2, \gamma=1\ \delta=2, m=9$ in the system (1.3). So we can obtain equilibrium $E_0 = (11.8, 12.93)$. Through calculating of the Mathematical software, we obtain $\omega_0 \approx 2.62, \tau_0 \approx 0.88$. By lemma 2.2 and above conclusion, we can obtain that equilibrium E_0 is stable if $\tau = 0.7 < \tau_0$, (see Figure 1). By contrast, the equilibrium E_0 is unstable if $\tau = 1.1 > \tau_0$, (see Figure 2). Besides, when $\tau_0 = 0.886$, the periodic solutions occur from the equilibrium E_0 (see Figure 3-4). However our analysis indicates that the dynamics of the comprehensive national strength model with time delay can be much more complicated than we may have expected. It is still interesting and inspiring to research.

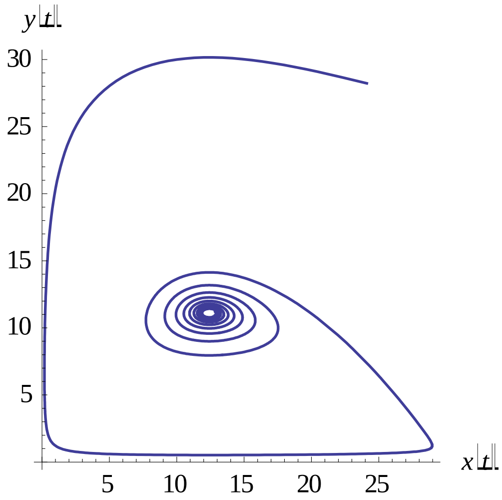

FIGURE 1 THE EQUILIBRIUM E_0 OF SYSTEM (1.3) IS STABLE WITH $\tau = 0.7 < \tau_0$

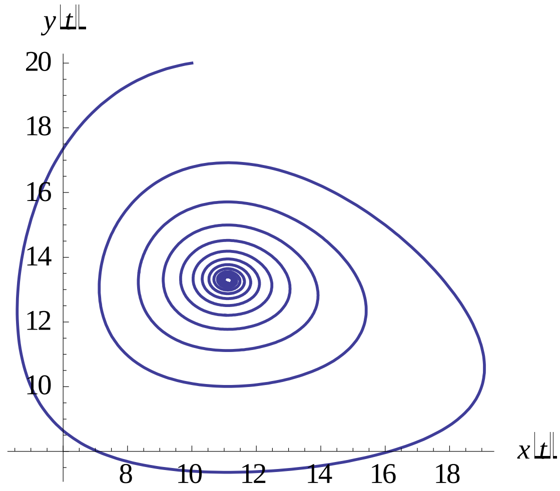

FIGURE 2 THE EQUILIBRIUM E_0 OF SYSTEM (1.3) IS STABLE WITH $\tau = 1.1 > \tau_0$

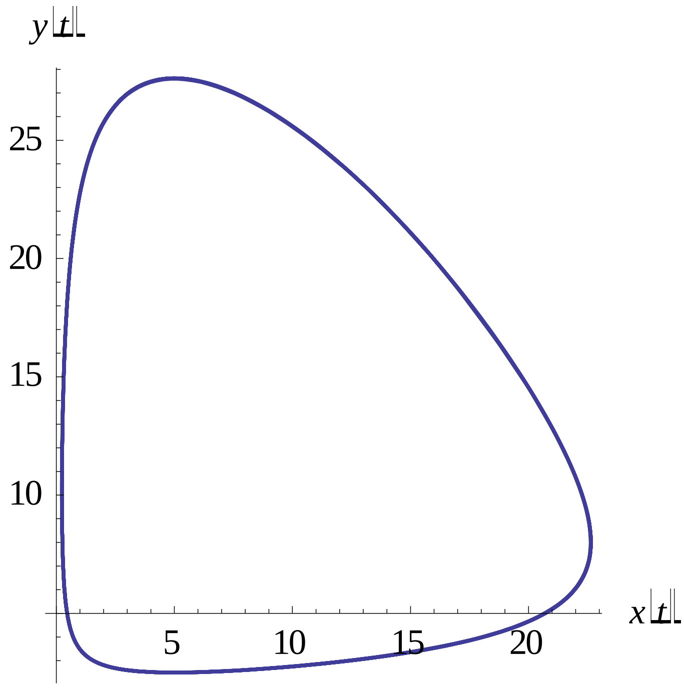

FIGURE 3 SYSTEM (1.3) PRODUCE THE PERIODIC SOLUTIONS WITH $\tau_0 = 0.886$

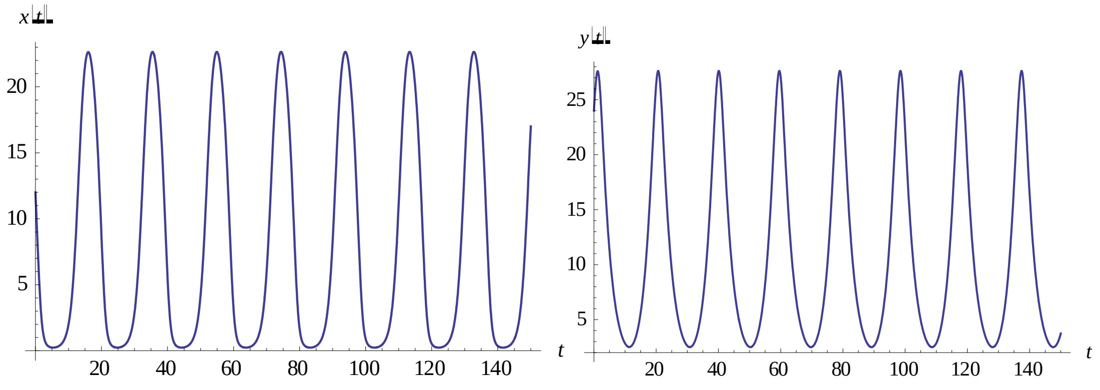

FIGURE 4 TIME SEQUENCE DIAGRAM OF FIGURE 3

V. CONCLUSION

These papers apply delay in the comprehensive national strength model, which show rich dynamics behavior. Different from previous studies, we added the influence of time delay feedback in the system (1.3). Dynamic behavior of comprehensive national strength model with time delay is analyzed by using the method of quantitative. When delay τ across a series of critical value, nonlinear dynamic system generate the Hopf bifurcation. In addition, by employing the normal form theory and center manifold method, we obtain some testable results on these issues. We use normative theory and center of popular theorem obtained the calculating method of the direction of Hopf bifurcation and stability of periodic solutions. Finally, the above theoretical analysis is verified by numerical simulation. The dynamic behavior of the comprehensive national strength model is rich. Many aspects is not mining, yet to be. References

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper.

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AUTHOR CONTRIBUTIONS

Conceived: Xiaohong Wang. Drawing graphics: Xiaohong Wang. Calculated: Xiaohong Wang. Modified: Yanhui Zhai. Wrote the paper: Xiaohong Wang.

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