Gap Functions and Error Bound to Set-Valued Variational Inequalities

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Abstract— *In this paper, the gap function for set-valued variational inequalities is introduced and the finiteness of the gap function is discussed. Furthermore, under μ-strongly monotone condition, we obtain error bounds for set-valued variational inequalities, i.e. upper estimates forthe distance to the solution set of the variational inequalities.*

Keywords— variational inequalities, gap functions, set-valued maps, error bounds.

I. INTRODUCTION

A vector variational inequality (for short,VVI) in a finite-dimensional Euclidean space was introduced first by Giannessi [1] in 1980.In the recent years, variational inequalities have become a very popular field of research in optimization theory. From the computational point of view, one important research direction in variational inequalities is the study of gap functions. One advantage of the introduction of gap functions in variational inequality is that variational inequalities can be transformed into optimization problem. Thus, powerful optimization solution methods and algorithms can be applied to find solutions of variational inequalities. Meanwhile, the gap functions can be used to devise error bounds for variational inequalities. There have been many results regarding gap functions and error bounds for classical variational inequalities see[2-5].

In this paper, we are interested in studying variational inequalities with set-valued maps. The solution of the variational inequalities with set-valued maps is a natural extension of the classic generalized variational inequalities studied in[6,7]. These kind of variational inequalities arise as generalization of optimality conditions for a constrained optimization problem with non-smooth objective function. See for example[8], where some equivalence of some particular set-valued VVI and a nondifferentiable and nonconvex vector optimization problem is established. Our aim in this article is to construct gap functions which can be used to devise error bounds for variational inequalities with set-valued maps.

Given a set-valued map $T: R^n \Rightarrow R^n$ and a nonempty and convex subset C of R^n, then $VI(T,C)$ consists in finding $\bar{x} \in C$ and $\bar{t} \in T(\bar{x})$ such that

$$\langle \bar{t}, y-\bar{x} \rangle \geq 0, \qquad \forall y \in C. \tag{1.1}$$

This problem is also referred to as Stampacchiavariational inequality defined by T and C .on the vector $y \in C$, this leads us to define what is called the weak Stampacchiavariational inequality,

denoted by $VI_w(T,C)$ which consists in finding $\bar{x} \in C$ such that, $\forall y \in C, \exists \bar{t}(y) \in T(\bar{x})$ satisfying

$$\langle \bar{t}(y), y-\bar{x} \rangle \geq 0, \qquad \forall y \in C \tag{1.2}$$

We shall denote by $S(T,C)$ the solution set of the $VI(T,C)$, that is

$$\bar{x} \in S(T,C) \Leftrightarrow \exists \bar{x}^* \in T(\bar{x}) \text{ such that } \langle \bar{x}^*, y-\bar{x} \rangle \geq 0, \quad \forall y \in C,$$

while $S_w(T,C)$ stands for the solution set of the $VI_w(T,C)$:

$$\bar{x} \in S_w(T,C) \Leftrightarrow \forall y \in C, \exists \bar{x}_y^* \in T(\bar{x}) \text{ such that } \langle \bar{x}_y^*, y-\bar{x} \rangle \geq 0.$$

Clearly one alwayshas $S(T,C) \subset S_w(T,C)$.

II. GAP FUNCTIONS

In this section,we introduce the concept of gap functions for $VI(T,C)$ and $VI_w(T,C)$.

Definition 2.1. $h: C \subset R^n \to R$ is said to be a gap functions of the $VI(T,C)$ or the

$VI_w(T,C)$ if

(i) $h(x) \geq 0, \quad \forall x \in C$;

(ii) $0 = h(\bar{x})$ if and only if $\bar{x}$ is a solution of the $VI(T,C)$ or the $VI_w(T,C)$.

Given a set-valued map $T: R^n \rightrightarrows R^n$, for any $\alpha > 0$ the normalized operator T_α defined by:

$$T_\alpha = \begin{cases} \{0\} & if\, 0 \in T(x) \\ \left\{ \dfrac{1}{\alpha \|x^*\|} x^* : \; x^* \in T(x) \right\}, & otherwise \end{cases}$$

In [9], Yang and Yao introduced the gap function and established necessary and sufficient conditions for the existence of a solution for variational inequalities with set-valued maps. In [10], D.Aussel. J.Dutta first introduced a simple gap function.

$$g_\alpha(x) := \sup_{y \in C} \inf_{x^* \in T_\alpha(x)} \langle x^*, x - y \rangle$$

In order to prove error bounds, it is important to know whether the gap function is finite or not. If C is bounded, the gap functions is finite. In order to overcome this boundedness

Assumption about C, D.Aussel. J.Dutta again introduced a class of gap functions.

$$g_{(\alpha,\beta)}(x) := \sup_{y \in C} [\inf_{x^* \in T_\alpha(x)} \langle x^*, x - y \rangle - \frac{1}{2\beta} \|x - y\|^2]$$

for any $\alpha > 0$ and $\beta > 0$.

Inspired and motivated by the above research work, in this paper ,the new gap functions for $VI(T,C)$ and $VI_w(T,C)$ are introduced.

Definition 2.2. An operator $T: R^n \rightrightarrows R^n$ is $\mu-$strongly monotone, if there exists $\mu > 0$ such that, for any $x, y \in R^n$ and any $x^* \in T(x), y^* \in T(y)$,one has

$$\langle y^* - x^*, y - x \rangle \geq \mu \|y - x\|^2. \tag{2.1}$$

Proposition 2.1. If $T: R^n \rightrightarrows R^n$ is $\mu-$strongly monotone with $\mu > 0$ and C is a nonempty convex set, then the $VI(T,C)$ and the $VI_w(T,C)$ have at most one solution.

Proof. If $\bar{x}$ and $\hat{x}$ all is solution of $VI(T,C)$, then $\exists \bar{t} \in T(\bar{x}), \hat{t} \in T(\hat{x})$ such that

$\langle \bar{t}, y - \bar{x} \rangle \geq 0, \; \forall y \in C$ and $\langle \hat{t}, y - \hat{x} \rangle \geq 0, \; \forall y \in C.$

There is a $y = \hat{x}$ such that $\langle \bar{t}, \hat{x} - \bar{x} \rangle \geq 0$ and a $y = \bar{x}$ such that $\langle \hat{t}, \bar{x} - \hat{x} \rangle \geq 0$.Combining above two inequalities, we have

$$\langle \bar{t} - \hat{t}, \bar{x} - \hat{x} \rangle \le 0.$$

Since $T: R^n \Rightarrow R^n$ is $\mu-$strongly monotone with $\mu > 0$, it follows that,

$$\langle \bar{t} - \hat{t}, \bar{x} - \hat{x} \rangle \ge \mu \|\bar{x} - \hat{x}\|^2 \ge 0,$$

and consequently, $\hat{x} = \bar{x}$.With the same proof as above one can obtain same consequence about $VI_w(T, C)$.

Let $\phi: R^n \times R^n \to R$ be a real-valued function. We make the following hypothesis.

Assumption 2.1.

(i) $\phi(x, y)$ is continuously differentiable in the second argument;

(ii) $\phi(x, y) \ge 0, \ \forall (x, y) \in R^n \times R^n$;

(iii) for all $x \in R^n$, $\phi(x, y)$ is uniformly strong convex in the second argument with respect to constant $\lambda > 0$; i.e., $\exists \lambda > 0$,for all $x \in R^n$

$$\phi(x, y_1) - \phi(x, y_2) \ge \langle \nabla_2 \phi(x, y_2), y_1 - y_2 \rangle + \lambda \|y_1 - y_2\|^2, \forall y_1, y_2 \in R^n;$$

(iv) $\phi(x, y) = 0$ if and only if $x = y$;

(v) $\nabla_2 \phi(x, y)$ is Lipschitz in the second argument with Lipschitz constant L_ϕ; i.e.,

$$\|\nabla_2 \phi(x, y_1) - \nabla_2 \phi(x, y_2)\| \le L_\phi \|y_1 - y_2\|, \quad \forall y_1, y_2 \in C.$$

The following examples show that Assumption 2.1 is satisfied by some functions.

Example 2.1. $\phi(x, y) = k\|x - y\|^2 \quad (k > 0).$

Example 2.2. $\phi(x, y) = \langle x - y, B(x)(x - y) \rangle,$

Where $B(x)$ is positive definite symmetric matrices and $B(x)$ is uniformly continuous differentiable.

Lemma 2.1. ([11]) Let Assumption 2.1 (i) - (iv) hold, then $\nabla_2 \phi(x, y) = 0$ if and only if $x = y$.

Proposition 2.2. Let Assumption 2.1 hold, then,

$$\lambda \|x - y\|^2 \le \phi(x, y) \le (L_\phi - \lambda)\|x - y\|^2, \quad \forall x, y \in C.$$

Proof. In Assumption 2.1 (iii), let $y_1 = y, y_2 = x$ and hence,

$$\phi(x, y) - \phi(x, x) \ge \langle \nabla_2 \phi(x, x), y - x \rangle + \lambda \|y - x\|^2.$$

By Assumption 2.1(iv), $\phi(x, x) = 0$ and by Lemma 2.1 , $\nabla_2 \phi(x, x) = 0$ .Hence,

$$\lambda \|x - y\|^2 \le \phi(x, y), \quad \forall x, y \in C.$$

In Assumption 2.1 (iii), let $y_1 = x, y_2 = y$ and hence,

$$\phi(x, x) - \phi(x, y) \ge \langle \nabla_2 \phi(x, y), x - y \rangle + \lambda \|x - y\|^2.$$

By Assumption 2.1(v) and Lemma 2.1 , we have that

$$\begin{aligned}-\phi(x,y) &\geq \langle \nabla_2\phi(x,y), x-y\rangle + \lambda\|x-y\|^2 \\ &\geq \langle \nabla_2\phi(x,y)-\nabla_2\phi(x,x), x-y\rangle + \lambda\|x-y\|^2 \\ &\geq -\|\nabla_2\phi(x,y)-\nabla_2\phi(x,x)\|\|x-y\| + \lambda\|x-y\|^2. \\ &\geq -L_\phi\|x-y\|^2 + \lambda\|x-y\|^2 = (\lambda - L_\phi)\|x-y\|^2\end{aligned}$$

Definition 2.3. A set-valued map $T: R^n \rightrightarrows R^n$ is said to be directionally closed if and only if there $\exists \alpha > 0$ such that for any $x \in R^n, T_\alpha(x)$ is closed.

Consider $VI(T,C)$ and $VI_w(T,C)$. We define the following function:

$$G_{(\alpha,\beta)}(x) := \sup_{y\in C}[\inf_{x^*\in T_\alpha(x)}\langle x^*, x-y\rangle - \beta\phi(x,y)]$$

for any $\alpha > 0$ and $\beta > 0$.

Proposition 2.3. Let the set-valued map $T: R^n \rightrightarrows R^n$ be directionally closed with nonempty values and let C be a closed convex subset. Then for any $\alpha > 0$ and $\beta > 0$, the function $G_{(\alpha,\beta)}(x)$ is a finite-valued function on R^n.

Proof. Let $\psi_\alpha(x,y) = \inf_{x^*\in T_\alpha(x)}\langle x^*, x-y\rangle$ and we immediately obtain

$$G_{(\alpha,\beta)}(x) = \sup_{y\in C}[\psi_\alpha(x,y) - \beta\phi(x,y)].$$

While x being fixed, the function $y \mapsto \psi_\alpha(x,y)$ is concave and further T_α has non-empty closed values when T is directionally closed with nonempty values. By T_α definition, we have that T_α is bounded, thus T_α is compact-valued. Hence the function $y \mapsto \psi_\alpha(x,y)$ is finite.

Moreover the function $y \mapsto -\psi_\alpha(x,y) + \beta\phi(x,y)$ is uniformly strong convex and thus it attains a unique minimum over C when C is closed.

Remark 2.1. For any $x \in R^n$, any $\alpha > 0$ and any $\beta > 0$ let us denote by $y_{(\alpha,\beta)}(x)$ the unique global minimum of the function $y \mapsto -\psi_\alpha(x,y) + \beta\phi(x,y)$ on the closed constraint set C.

Lemma 2.2. ([10]) Let T be any set-valued map from R^n to R^n and C is a nonempty subset of R^n. Then the following assertions are equivalent:

(i) $\bar{x} \in S(T,C)$;

(ii) there exists $\alpha > 0$, such that $\bar{x} \in S(T_\alpha, C)$;

(iii) for any $\alpha > 0$, $\bar{x} \in S(T_\alpha, C)$.

Theorem 2.1. Let the set-valued map $T: R^n \rightrightarrows R^n$ be directionally closed with nonempty values and let C be a closed convex subset. Then the function $G_{(\alpha,\beta)}(x)$ is a gap function for the $VI_w(T,C)$.

Proof. It is simple to observe that $G_{(\alpha,\beta)}(x) \geq 0$ for all $x \in C$ just by setting $x = y$.

Let us now assume that $G_{(\alpha,\beta)}(\bar{x})=0$, there $\bar{x}\in C$. i.e.,

$$G_{(\alpha,\beta)}(\bar{x})=\sup_{y\in C}[\inf_{x^*\in T_\alpha(x)}\langle x^*,\bar{x}-y\rangle-\beta\phi(\bar{x},y)]=0.$$

Thus,

$$\inf_{x^*\in T_\alpha(x)}\langle x^*,\bar{x}-y\rangle-\beta\phi(\bar{x},y)\leq 0,\ \ \forall y\in C.$$

By Proposition 2.2, we have

$$\phi(\bar{x},y)\leq(L_\phi-\lambda)\|\bar{x}-y\|^2,\quad \forall y\in C.$$

Hence

$$\inf_{x^*\in T_\alpha(x)}\langle x^*,\bar{x}-y\rangle\leq\beta\phi(\bar{x},y)\leq\beta(L_\phi-\lambda)\|\bar{x}-y\|^2,\ \ \forall y\in C.$$

Let us consider a fixed but arbitrary point $x_0\in C$. Consider the sequence of vector $\{y_k\}$ given as

$$y_k=\bar{x}+\frac{1}{k}(x_0-\bar{x}),\quad k\in N.$$

Since C is convex it is clear that $y_k\in C$ for each $k\in N$. Therefore , for any k,

$$\inf_{x^*\in T_\alpha(x)}\langle x^*,\bar{x}-y_k\rangle\leq\beta\phi(\bar{x},y_k)\leq\beta(L_\phi-\lambda)\|\bar{x}-y_k\|^2.$$

Combining above two inequalities, we have

$$\inf_{x^*\in T_\alpha(x)}\langle x^*,\bar{x}-x_0\rangle\leq\frac{1}{k}\beta(L_\phi-\lambda)\|\bar{x}-x_0\|^2.$$

As $k\to\infty$ we have

$$\inf_{x^*\in T_\alpha(x)}\langle x^*,\bar{x}-x_0\rangle\leq 0.$$

Since T_α is compact-valued, there exists $x_0^*\in T_\alpha(\bar{x})$ such that

$$\langle x_0^*,\bar{x}-x_0\rangle=\inf_{x^*\in T_\alpha(x)}\langle x^*,\bar{x}-x_0\rangle\leq 0.$$

Hence

$$\langle x_0^*,x_0-\bar{x}\rangle\geq 0.$$

Since $x_0\in C$ is arbitrary, it is clear that $\bar{x}\in S_w(T,C)$.

Conversely, let us now assume $\bar{x}\in S_w(T,C)$. By Lemma 2.2, we obtain $\bar{x}\in S_w(T_\alpha,C)$, that is, for $\forall y\in C,\exists\bar{x}_y^*\in T_\alpha(\bar{x})$ such that

$$\langle\bar{x}_y^*,y-\bar{x}\rangle\geq 0.$$

This immediately implies that

$$G_{(\alpha,\beta)}(x) \le 0.$$

This completes the proof.

Theorem 2.2. Let the set-valued map $T : R^n \rightrightarrows R^n$ be directionally closed with nonempty values and let C be a closed convex subset. Then $\bar{x} \in S_w(T,C)$ if and only if $\bar{x} = y_{(\alpha,\beta)}(\bar{x})$ for any $\alpha > 0$ and $\beta > 0$.

Proof. From the definition of $y_{(\alpha,\beta)}(x)$, if $\bar{x} = y_{(\alpha,\beta)}(\bar{x})$, we immediately obtain

$$G_{(\alpha,\beta)}(\bar{x}) = \sup_{y\in C}[\psi_\alpha(\bar{x}, y) - \beta\phi(\bar{x}, y)] = \psi_\alpha(\bar{x},\bar{x}) - \beta\phi(\bar{x},\bar{x}) = 0.$$

By Theorem 2.1 , we have $\bar{x} \in S_w(T,C)$.Now let us suppose, for a contradiction, that $\bar{x} \in S_w(T,C)$ and that $\bar{x} \ne y_{(\alpha,\beta)}(\bar{x})$.By $\bar{x} \in S_w(T,C)$, for each $y \in C$,there exists $\bar{x}_y^* \in T_\alpha(\bar{x})$ such that $\langle \bar{x}_y^*, y - \bar{x}\rangle \ge 0.$ Hence, for each $y \in C$, $\psi_\alpha(\bar{x}, y) \le 0$. By Assumption 2.1(iv), while $\bar{x} \ne y_{(\alpha,\beta)}(\bar{x})$ we have $\phi(\bar{x}, y_{(\alpha,\beta)}(\bar{x}) < 0$. Since

$$G_{(\alpha,\beta)}(\bar{x}) = \psi_\alpha(\bar{x}, y_{(\alpha,\beta)}(\bar{x})) - \beta\phi(\bar{x}, y_{(\alpha,\beta)}(\bar{x})),$$

thus we have $G_{(\alpha,\beta)}(\bar{x}) < 0$, which is a contradiction with Theorem 2.1.

Remark 2.2. Theorem 2.2 provide an interesting fixed point characterization of the solutions of $VI_w(T,C)$.

3.Error Bounds

Theorem 3.1. Suppose that $T : R^n \rightrightarrows R^n$ is $\mu-$strongly monotone with $\mu > 0$ and let $\bar{x} \in C$ be the unique minimum of $VI_w(T,C)$.Then, for any $\alpha > 0$ and $\beta > 0$, such that $\mu - \beta(L_\phi - \lambda) > 0$, one has for any $x \in C$,

$$\|x - \bar{x}\| \le \sqrt{\frac{1}{\mu - \beta(L_\phi - \lambda)} G_{(\alpha,\beta)}(x)}.$$

Proof. By $\bar{x} \in S_w(T,C)$, for each $y \in C$,there exists $\bar{x}_y^* \in T_\alpha(\bar{x})$ such that

$$\langle \bar{x}_y^*, y - \bar{x}\rangle \ge 0.$$

Thus, for $\forall x^* \in T(x)$, we have

$$\langle x^*, x - \bar{x}\rangle \ge \langle x^* - \bar{x}_y^*, x - \bar{x}\rangle \ge \mu\|x - \bar{x}\|^2.$$

By Proposition 2.2, we have

$$\phi(\bar{x}, y) \le (L_\phi - \lambda)\|\bar{x} - y\|^2, \quad \forall y \in C.$$

Hence,

$$\begin{aligned} G_{(\alpha,\beta)}(x) &= \sup_{y\in C}[\inf_{x^*\in T_\alpha(x)} \langle x^*, x - y\rangle - \beta\phi(x, y)] \\ &\ge \inf_{x^*\in T_\alpha(x)} \langle x^*, x - \bar{x}\rangle - \beta\phi(x, \bar{x}) \\ &\ge \mu\|x - \bar{x}\|^2 - \beta(L_\phi - \lambda)\|x - \bar{x}\|^2 = [\mu - \beta(L_\phi - \lambda)]\|x - \bar{x}\|^2. \end{aligned}$$

Therefore, we have

$$\|x-\bar{x}\| \le \sqrt{\frac{1}{\mu-\beta(L_\phi-\lambda)} G_{(\alpha,\beta)}(x)}\,.$$

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REFERENCES

[1] F.GIANNESSI, Theorems of Alternative, Quadratic Programmes and Complementarity Problems, in: R. W. Cottle, F.Giannessi, J.L.Lions(Eds.), Variational Inequalities and Complementarity Problems, Wiley, Chichester, 1980,pp.151-186.

[2] Auslender A. Optimization Methods Numeriques. Masson: Paris,1976.

[3] Fukushima M. Equivalent Differentiate Optimization Problems and Descent Methods for Asymmetric Variational Inequality Problems. Mathematical Programming, 1992,53:99-110.

[4] Solodov M V. Merit Functions and Error Bounds for Generalized Variational Inequalities. Journal of Mathematical Analysis and Applications, 2003,287:405-414.

[5] Noor M A. Merit Functions for General Variational Inequalities. Journal of Mathematical Analysis and Applications, 2006,316:736-752.

[6] BROWDER, F.E.,HESS,P.,Nonlinear Mappings of Monotone Type in Banach Spaces,Journal of Functional Analysis. Vol.11,pp.251-294,1972.

[7] KRAVVARITTIS, D., Nonlinear Equations and Inequalities in Banach Spaces, Journal of Mathematical Analysis and Applications,Vol.67,pp.205-214,1979.

[8] ANSARI,Q.H., and YAO,J.C., Nondifferentiable and Nonconvex Optimization Problems, Journal of Optimization Theory and Applications, Vol.106, pp.475-488,2000.

[9] X. Q. YANG, J. C. YAO, Gap Functions and Existence of Solutions to Set-Valued Vector Variational Inequalities, Journal of Optimization Theory and Applications, Vol.115, pp.407-417,2002.

[10] D.Aussel.J.Dutta, On Gap Functions Multivalued StampacchiaVariational Inequalities,Journal of Optimization Theory and Applications, Vol.149, pp.513-527,2011.

[11] Yamashita N., Taji K., Fukushima M. Unconstrained Optimization Reformulations of Variational Inequality Problems, Journal of Optimization Theory and Applications, Vol.92, pp.439-456,1997.