Numerical Treatment of the Stochastic Advection-Diffusion Equation Using the Spectral Stochastic Techniques

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Abstract— *Stochastic advection diffusion equation (SADE) with multiplicative stochastic input is a practical mathematical model for different physical phenomena. In this paper, SADE will be studied using two spectral stochastic techniques. The first is the Wiener chaos expansion (WCE) technique and the second is the Wiener-Hermite expansion with perturbation (WHEP) technique. These techniques convert the SADE into a system of deterministic partial differential equation (DPDE) that can be solved using a deterministic numerical method which is suitable for the periodic boundary conditions. Convergence analysis is discussed and some of the second order moments are compared. The numerical results demonstrate the efficiency of both techniques. The WCE technique is more accuracy than the WHEP technique. The diffusion and advection coefficient and the intensity of Gaussian white noise play important roles in the SADE solution. The study shows that the WCE technique is more practical to get the closed form mean solution while the WHEP technique gets the mean solution in the form of an infinite series.*

Keywords— Advection diffusion equation, Multiplicative random input, Wiener-Hermite expansion, Wiener-Chaos expansion, Periodic boundary conditions.

I. INTRODUCTION

Numerical models have received a great attention in sciences and engineering in the recent years for modeling the differential equations. These models work on reducing the cost and time of computation especially for the physical phenomena that contain uncertain input. This phenomena can be studied by converting it into mathematical models of stochastic differential equations (SDE) and we can use the numerical methods [1,2] for overcoming these problems.

Numerical methods have been developed for simulating SDE such as moment equations, probability density method [3], etc. These methods are complicated in solving the nonlinear SDEs so the spectral decomposition techniques have received much attention in the recent years. The spectral decomposition technique was first suggested by the great mathematician Norbert Wiener [4]. Wiener constructed an orthonormal random basis for expanding homogeneous chaos depending on white noise, and used it to study problems in statistical mechanics [5]. The Hermite polynomial has been used to obtain the solution of SDE. Mecham et al. [6] suggested the Wiener-Hermite expansion to study turbulence solution of Burger equation. In nonlinear stochastic differential equations, there exist always difficulties of solving the resultant set of deterministic integro-differential equations. The deterministic integro-differential equations got from the applications of a set of comprehensive averages on the stochastic integro-differential equation obtained after the direct application of WHE. Many authors introduced different methods to face these obstacles. Among them, the WHEP technique [7] was introduced using the perturbation technique to solve perturbed nonlinear problems. M. El-Tawil and his co-workers [7-11] used the WHE together with the perturbation theory (WHEP technique) to solve a perturbed nonlinear stochastic differential equation. The WHEP technique is generalized to handle n^{th} order polynomial nonlinearities, general order of WHE and general number of corrections [8].

Cameron and martin [12] was developed a more explicit and intuitive formulation for the Hermite polynomial, which was called the Wiener-Chaos expansion. Their development is based on an explicit discretization of the white noise process through its Fourier expansion. This approach is much easier to understand and more convenient to use, and hence replaced Wiener's original formulation. Fourier chaos expansion has become a useful tool in stochastic analysis involving Brownian motion [13]. Rozovskii et al. [14-16] derived Wiener chaos propagator equations for several important the stochastic partial differential equations (SPDEs) driven by Brownian motion forcing. Lototsky et al. [17, 18] proposed a new numerical method for solving the Zakai equation based on its Wiener chaos expansion. Using Fourier-Hermite expansion for modeling non-Gaussian processes is also investigated [19, 20]. Babuska et al. [21], Schwab et al. [22] and Keese el al. [23] developed and generalized Ghanem's approach for solving stochastic elliptic equations. Xiu and Karniadakis [24] generalized the Hermite polynomial expansion and used it to study flow-structure interactions. Zhang et al. [25] combined moment perturbation method with polynomial chaos expansion, and used it to study the saturation flows in heterogeneous porous media.

The main goal of this paper is to use two stochastic spectral techniques, WCE and WHEP for solving the stochastic advection diffusion equation with multiplicative white noise and periodic boundary conditions. The two techniques convert SPDE into a system of DPDE. The DPDE can be solved using a proposed Eigen function expansion in both cases of WCE and WHEP techniques. The results will be studied through the mean and variance solutions.

This paper is organized as follows:

The formulation of the SADE is outlined in section 2. The WCE technique is explained in section 3. In section 4, the algorithm of WHEP technique is introduced. In section 5 and 6, we apply the WCE and WHEP techniques respectively; the convergence analysis of the WCE is studied. The proposed method for solving the resulting DPDEs; the numerical solutions of the WCE and WHEP techniques are introduced in section 7. The comparison and discussion of the results of the two techniques are in section 8. Finally, the conclusions are given in Section 9.

II. STOCHASTIC ADVECTION DIFFUSION EQUATION

SADE represents the transporting that occurs in fluids through the combination of advection process and diffusion process .Consider the SADE in Stratonovich sense with multiplicative stochastic force described by white noise and periodic boundary conditions as [27]:

$$\begin{aligned} &du(x,t) = \mu \frac{\partial^2}{\partial x^2} u(x,t)dt + \sigma \frac{\partial}{\partial x} u(x,t) \circ dW_t, \quad t>0, \\ &u(0,x) = \sin(x), \quad x \in (0,2\pi), \end{aligned} \tag{2.1}$$

which can be written in the Ito sense as:

$$\begin{aligned} &du(x,t) = \gamma \frac{\partial^2}{\partial x^2} u(x,t)dt + \sigma \frac{\partial}{\partial x} u(x,t) dW_t, \quad t>0, \\ &u(0,x) = \sin(x), \quad x \in (0,2\pi), \end{aligned} \tag{2.2}$$

where $u(x,t)$ represents the concentration of mass transfer; $\mu \geq 0$ represents the diffusion coefficient; $\sigma > 0$, $\gamma = \mu + \frac{\sigma^2}{2}$ and W_t is the one dimensional Brownian motion.

This model has exact solution [27] as:

$$u(x,t) = e^{-\mu t} \sin(x + \sigma W(t)). \tag{2.3}$$

The first order moment $E[u(x,t)]$ and the second order moment $E[u(x,t)]^2$ are:

$$\begin{aligned} &E[u(x,t)] = e^{-\gamma t} \sin(x), \\ &E[u(x,t)]^2 = e^{-2\mu t}\left(\frac{1}{2} - \frac{1}{2} e^{-2\sigma^2 t} \cos(2x)\right), \end{aligned} \tag{2.4}$$

So the solution admits the Wiener chaos expansion because of finite second moment.

III. WIENER CHAOS EXPANSION TECHNIQUE

For SPDE with random force in form of Brownian motion (BM) and for fixed time $T > 0$, we consider an orthnormal basis in Hilbert space $L^2([0,T])$ to be the trigonometric functions [16]:

$$m_1(t)=\sqrt{\frac{1}{T}},\quad m_i(t)=\sqrt{\frac{2}{T}}\cos\left(\frac{(i-1)\pi t}{T}\right),\quad i>1,\ 0\le t\le T.$$

Also define the standard Gaussian random variables ζ_i (GRVs) and the Brownian motion $W_t=W(t)$ as follows:

$$\zeta_i=\int_0^t m_i(s)dW_S,\qquad W(t)=\sum_{i=1}^{\infty}\zeta_i\int_0^t m_i(s)ds. \tag{3.1}$$

The convergence in mean square sense [26] in the interval $[0,T]$ is:

$$E[W(t)-\sum_{i=1}^{N}\zeta_i\int_0^t m_i(\tau)d\tau]^2<\frac{T}{\pi N},\quad 0\le t\le T\ .$$

According to [12], the solution function $u(x,t)$ can be expanded as follows:

$$u(x,t)=\sum_{\alpha\in\Im}u_\alpha(x,t)T_\alpha(\zeta),\quad u_\alpha=E[u\,T_\alpha(\zeta)]=E[u\,T_\alpha(t)], \tag{3.2}$$

where the multi-indices $\Im=\{\alpha=(\alpha_i,i\ge1),\ \alpha_i\in\{0,1,2,...\};|\alpha|=\sum_{i=1}^{\infty}\alpha_i<\infty\}$ and $T_\alpha(\zeta)=\prod_{i=1}^{\infty}H_{\alpha_i}(\zeta_i)$ where $H_n(x)$ is the normalized n^{th} order Hermite polynomial and T_α are called Wick polynomials of order α . Also $T_\alpha(t)$ is the Wick polynomials filtered by the σ-algebra F_t^W, Recall that $T_\alpha(t)$ is a martingale and satisfies the differential equation [26]:

$$dT_\alpha(t)=\sum_{i\ge1}\sqrt{\alpha_i}\,m_i(t)T_{\bar{\alpha}(i)}\,dW_t, \tag{3.3}$$

where multi-index $\bar{\alpha}_i(j)=\begin{cases}\alpha_j, & j\ne i,\\ \alpha_{i-1}, & j=i.\end{cases}$.

Wick polynomials form a complete orthonormal basis in the Hilbet space. The expectation of two wich polynomials T_α,T_β is $E[T_\alpha T_\beta]=\delta_{\alpha,\beta}$.

Truncating the expansion (3.2) up to polynomial of order N and using only K GRVs retains $\frac{(K+N)!}{N!(K)!}$ coefficients [26]. The truncated multi-indices will be $\Im_{K,N}=\{\alpha=(\alpha_1,\ldots\alpha_K),\ \alpha_i\in\{0,1,2,...\};|\alpha|=\sum_{i=1}^{K}\alpha_i\le N\}$. Then the truncated WCE can be denoted as:

$$u_{K,N}(x,t)=\sum_{\alpha\in\Im_{K,N}}u_\alpha(x,t)T_\alpha(\zeta). \tag{3.4}$$

The mean and variance of the truncate solution function $u_{K,N}(x,t)$ are computed as:

$$E[u_{K,N}(x,t;\omega)]=u_0(x,t),$$

$$\mathrm{var}[u_{K,N}(x,t;\omega)]=\sum_{\alpha\in\Im_{K,N},\alpha\ne0}\left|u_\alpha(x,t)\right|^2 .$$

IV. WIENER HERMITE EXPANSION

As a consequence of the completeness of the Wiener-Hermite set [28] any arbitrary stochastic process can be expanded in terms of the Wiener-Hermite polynomial set $H^{(n)}(t_1,...,t_n)$. This expansion converges to the original stochastic process with probability one.

The stochastic solution process $u(x,t;w)$ can be expanded as, [7]:

$$u(x,t;\omega) = u^{(0)}(x,t) + \sum_{k=1}^{\infty} \int_{R^k} u^{(k)}(x,t;t_1,...,t_k) H^{(k)}(t_1,...,t_k)\, d\tau_k \,, \tag{4.1}$$

where $d\tau_k = dt_1 dt_2 ....dt_k$ and $\int_{R^k}$ is a k-dimensional integral over the disposable variables $t_1, t_2, ..., t_k$. The functional $H^{(n)}(t_1, t_2, ...t_n)$ is the n[th] order Wiener-Hermite time independent functional and $u^{(i)}(x,t;t_1,t_2,...t_i);\, i \ge 0$ are the deterministic kernel of the WHE.

The Wiener-Hermite functionals $H^{(n)}$ form a complete set with $H^{(0)} = 1$ and $H^{(1)}(t) = \frac{dW}{dt}$ is the white noise. The $H^{(n)}$ functions are statistically orthnormal, i.e.

$$\begin{aligned} E\left[H^{(i)}\right] &= 0 \quad \forall i \ge 1, \\ E\left[H^{(i)} H^{(j)}\right] &= 0 \quad \forall i \ne j. \end{aligned} \tag{4.2}$$

The solution will be practically truncated with $(m+1)$ terms and the expectation and variance of the truncated solution will be:

$$\begin{aligned} E[u(x,t;\omega)] &= u^{(0)}(x,t), \\ \mathrm{var}[u(x,t;\omega)] &= \sum_{k=1}^{m} k! \int_{R^k} \left(u^{(k)}(x,t;t_1,...,t_k)\right)^2 d\tau_k. \end{aligned}$$

In the nonlinear SPDE or the multiplicative SPDE, it is difficult to solve the deterministic differential-integral equation system of the kernels results from the application of the WHE. This difficulty is due to the resulting system is a coupled differential-integral system and we can overcome this by using the perturbation technique.

In the perturbation technique the solution is a power series of small parameter σ. Set of simple equations are expanded as [29]:

$$u^{(k)} = \sum_{i=0}^{NC} \sigma^i u_i^{(k)},\ k \ge 0\,, \tag{4.3}$$

where NC is the number of corrections. For m order WHE, the statistical properties of the relatively solution will be calculated as:

$$\begin{aligned} E[u(x,t;\omega)] &= \sum_{i=0}^{NC} \sigma^i u_i^{(0)}(x,t), \\ \mathrm{var}[u(x,t;\omega)] &= \sum_{k=1}^{m} k! \int_{R^k} \left(\sum_{i=0}^{NC} \sigma^i u_i^{(k)}(x,t;t_1,...,t_k) \right)^2 d\tau_k. \end{aligned} \tag{4.4}$$

V. APPLICATION OF WCE

To get the WCE of SADE (2.2); consider the differential form:

$$d[uT_\alpha] = du.T_\alpha + u.dT_\alpha + du.dT_\alpha,$$

using equations (2.2) and (3.3) to get:

$$\begin{aligned} d[uT_\alpha] = &\left(\gamma \frac{\partial^2}{\partial x^2} u(x,t)dt + \sigma \frac{\partial}{\partial x} u(x,t)dW_t\right)T_\alpha + u\left(\sum_{i\ge1}\sqrt{\alpha_i}\, m_i(t) T_{\bar{\alpha}(i)}\, dW_t\right) + \\ &\left(\gamma \frac{\partial^2}{\partial x^2} u(x,t)dt + \sigma \frac{\partial}{\partial x} u(x,t)dW_t\right).\left(\sum_{i\ge1}\sqrt{\alpha_i}\, m_i(t) T_{\bar{\alpha}(i)}\, dW_t\right). \end{aligned} \tag{5.1}$$

Taking the expectation for both sides of (5.1), the terms involving Ito integrals will disappear since they are mean zero. Then we will get,

$$d[u_\alpha(x,t)] = \left(\gamma \frac{\partial^2}{\partial x^2} u_\alpha(x,t)dt\right) + \sigma\left(\sum_{i\ge1}\sqrt{\alpha_i}\, m_i(t) \frac{\partial}{\partial x} u_{\bar{\alpha}(i)}(x,t)\right), \tag{5.2}$$

with the initial condition $u_\alpha(x,0) = \begin{cases} \sin(x), & \alpha=0, \\ 0, & \alpha\neq0. \end{cases}$.

Theorem:

The SADE (2.1) which has exact solution $u(x,t)$ and the truncated solution $u_{K,N}(x,t)$; the estimated error will take the form:

$$E\left|u(x,t) - u_{K,N}(x,t)\right| \le Ce^{-\mu t}\left(\frac{\sigma T}{K} + \frac{(\sigma\ T)^{N+1}}{(N+1)!}\right).$$

Proof:

according to [27], since $u(x,t) = \theta(x + \sigma W(t), t),$ and $W(t) = \sum_{i=1}^{\infty} \zeta_i \int_0^t m_i(s)ds = \frac{t}{\sqrt{T}}\zeta_1 + \sum_{i=2}^{\infty}\zeta_i \frac{\sqrt{2T}}{(i-1)\pi}\sin\left(\frac{(i-1)\pi t}{T}\right)$, let $W(t) = W_K + W_S$,

$W_K = \frac{t}{\sqrt{T}}\zeta_1 + \sum_{i=2}^{K}\zeta_i \frac{\sqrt{2T}}{(i-1)\pi}\sin\left(\frac{(i-1)\pi t}{T}\right)$ and $W_S = \sum_{i=K+1}^{\infty}\zeta_i \frac{\sqrt{2T}}{(i-1)\pi}\sin\left(\frac{(i-1)\pi t}{T}\right)$ where W_K and W_S are orthogonal. After that we can expand the solution $\theta(x + \sigma W_K + \sigma W_S, t)$ by Taylor's series with respect to W_K and W_S respectively, to get:

$$u(x,t) = \theta(x,t) + \sum_{m=1}^{N}\frac{(\sigma W_K)^m}{m!}\frac{\partial^m}{\partial x^m}\theta(x,t) + \frac{\partial}{\partial x}\theta(x + \sigma W_K + \eta_1, t)\sigma w_S + \frac{(\sigma W_K)^{N+1}}{(\mathrm{N}+1)!}\frac{\partial^{N+1}}{\partial x^{N+1}}\theta(x + \eta_2, t).$$

Truncating the solution with respect to order N and K GRVs to get:

$$u_{K,N}(x,t) = \theta(x,t) + \sum_{m=1}^{N}\frac{(\sigma W_K)^m}{m!}\frac{\partial^m}{\partial x}\theta(x,t),$$

then we have $E\left|u(x,t) - u_{K,N}(x,t)\right| \le C_1\sigma E\left|W_S^2\right| + C_{N+1}\frac{\sigma^{N+1}}{(N+1)!}E\left|W_K^{2N+2}\right|,$

where $C_n = \sup_x\left|\frac{\partial}{\partial x}\theta(x,t)\right|,$

then $E\left|W_S^2\right| \le \frac{2T}{\pi^2}\sum_{i=K+1}^{\infty}\frac{1}{(K-1)^2}\sin^2(\frac{(i-1)\pi t}{T}) < \frac{2T}{\pi^2 K}$,

$E\left|W_K^{2N+2}\right| = (E\left|W_K^2\right|)^{N+1}(2N+1)!! < (E\left|W^2\right|)^{N+1}(2N+1)!!,$

where $(2N+1)!! = (2N+1)(2N-1)........1,$

then,

$$E\left|u(x,t)-u_{K,N}(x,t)\right| \le Ce^{-\mu t}(\frac{\sigma T}{K}+\frac{(\sigma\ T)^{N+1}}{(N+1)!}). \tag{5.3}$$

This means that, the error between the approximate solution and the exact solution decays by increasing the number K of GRVs, the order of polynomial chaos N and diffusion coefficient μ. Also decreasing the advection coefficient σ and the time interval T increases the convergence between the exact and the WCE approximation.

VI. APPLICATION OF WHE

The first two terms in expansion (4.1) are the Gaussian part of the solution. This part is not sufficient for the accurate solution of SADE. The second order WHE is applied to the SADE (2.2) to get:

$$\frac{\partial}{\partial t}u^{(0)}(x,t)+\int_0^t\frac{\partial}{\partial t}u^{(1)}(x,t;t_1)H^{(1)}(t_1)dt_1+\int_0^t\int_0^t\frac{\partial}{\partial t}u^{(2)}(x,t;t_1,t_2)H^{(2)}(t_1,t_2)dt_1dt_2 =$$
$$\gamma\left[\frac{\partial^2}{\partial x^2}u^{(0)}(x,t)+\int_0^t\frac{\partial^2}{\partial x^2}u^{(1)}(x,t;t_1)H^{(1)}(t_1)dt_1+\int_0^t\int_0^t\frac{\partial^2}{\partial x^2}u^{(2)}(x,t;t_1,t_2)H^{(2)}(t_1,t_2)dt_1dt_2\right]+$$
$$\sigma n(t)\left[\frac{\partial}{\partial x}u^{(0)}(x,t)+\int_0^t\frac{\partial}{\partial x}u^{(1)}(x,t;t_1)H^{(1)}(t_1)dt_1+\int_0^t\int_0^t\frac{\partial}{\partial x}u^{(2)}(x,t;t_1,t_2)H^{(2)}(t_1,t_2)dt_1dt_2\right],$$
$$u^{(0)}(x,0)=\sin(x),\ u^{(1)}(x,0;t_1)=u^{(2)}(x,0;t_1,t_2)=0. \tag{6.1}$$

Multiplying both sides of (6.1) with $H^{(0)}, H^{(1)}(t_1)$ and $H^{(2)}(t_1,t_2)$ respectively and taking the expectation to get:

$$\frac{\partial}{\partial t}u^{(0)}(x,t)=\gamma\frac{\partial^2}{\partial x^2}u^{(0)}(x,t)+\sigma\frac{\partial}{\partial x}u^{(1)}(x,t;t). \tag{6.2}$$

$$\frac{\partial}{\partial t}u^{(1)}(x,t;t_1)=\gamma\frac{\partial^2}{\partial x^2}u^{(1)}(x,t;t_1)+\sigma\frac{\partial}{\partial x}u^{(0)}(x,t)\delta(t-t_1)+2\sigma\frac{\partial}{\partial x}u^{(2)}(x,t;t,t_1). \tag{6.3}$$

$$\frac{\partial}{\partial t}u^{(2)}(x,t;t_1,t_2)=\gamma\frac{\partial^2}{\partial x^2}u^{(2)}(x,t;t_1,t_2)+\frac{1}{2}\sigma\frac{\partial}{\partial x}u^{(1)}(x,t;t_1)\delta(t-t_2)+\frac{1}{2}\sigma\frac{\partial}{\partial x}u^{(1)}(x,t;t_2)\delta(t-t_1). \tag{6.4}$$

The deterministic systems appear when applying WCE and WHE are coupled integro-differential system of equations that are not easy to solve. In the following section, we will suggest a numerical technique to solve the deterministic systems (5.2) and (6.2-6.4) with periodic boundary conditions.

VII. THE NUMERICAL TECHNIQUE USING THE PRINCIPAL OF EIGEN FUNCTION EXPANSION

We introduce a numerical technique for solving the DPDE resulting from the application of the WCE and WHEP techniques. Here we implement the usual Eigen function [30] to be suitable for solving the resulting system of differential equations with periodic boundary conditions. The solution $u(x,t)$ can be written as an Eigen function expansion as:

$$u(x,t)=\sum_{n=0}^{\infty}\left[C_n^1\, e^{-an^2 t}+\int_0^t e^{-an^2(t-s)}F_n^1(s)\,ds\right]\cos(nx)+\sum_{n=1}^{\infty}\left[C_n^2\, e^{-an^2 t}+\int_0^t e^{-an^2(t-s)}F_n^2(s)\,ds\right]\sin(nx), \tag{7.1}$$

where,

$$C_0^1=\frac{1}{L}\int_0^L f(x)\,dx, \qquad F_0^1=\frac{1}{L}\int_0^L F(x,t)\,dx,$$
$$C_n^1=\frac{2}{L}\int_0^L f(x)\cos(nx)\,dx, \quad , \qquad F_n^1=\frac{2}{L}\int_0^L F(x,t)\cos(nx)\,dx, \tag{7.2}$$
$$C_n^2=\frac{2}{L}\int_0^L f(x)\sin(nx)\,dx, \qquad F_n^2=\frac{2}{L}\int_0^L F(x,t)\sin(nx)\,dx.$$

The proof of formulae (7.1) is explained in Appendix A.

The system of propagators (5.2) can be re-written using the Eigen function expansion (7.1) as:

$$u_\alpha(x,t)=\sum_{n=0}^{\infty}\left[C_n^1\, e^{-an^2 t}+\int_0^t e^{-an^2(t-s)}F_n^1(s)\,ds\right]\cos(nx)+\sum_{n=1}^{\infty}\left[C_n^2\, e^{-an^2 t}+\int_0^t e^{-an^2(t-s)}F_n^2(s)\,ds\right]\sin(nx). \tag{7.3}$$

For $|\alpha|=0$ we get:

$$\frac{\partial}{\partial t}u(x,t)=\gamma\frac{\partial^2}{\partial x^2}u(x,t), \quad u_0(x,0)=\sin(x),$$

which results in:

$$u_\alpha(x,t)=e^{-\gamma t}\sin(x). \tag{7.4}$$

For $|\alpha|=1,\ \alpha_1=1,\ \alpha_i=0$ we get:

$$\frac{\partial}{\partial t}u(x,t)=\gamma\frac{\partial^2}{\partial x^2}u(x,t)+\sigma m_1(t)\frac{\partial}{\partial x}u_{|\alpha|=0}(x,t), \quad u_\alpha(x,0)=0,$$

which results in:

$$u_\alpha(x,t)=\frac{\sigma t}{\sqrt{T}}e^{-\gamma t}\cos(x). \tag{7.5}$$

For $|\alpha|=1,\ \alpha_i=1,\ \alpha_1=0,\ i>1$ we get:

$$\frac{\partial}{\partial t}u(x,t)=\gamma\frac{\partial^2}{\partial x^2}u(x,t)+\sigma m_i(t)\frac{\partial}{\partial x}u_{|\alpha|=0}(x,t), \quad u_\alpha(x,0)=0,$$

which results in:

$$u_\alpha(x,t)=\frac{\sigma\sqrt{2}}{\pi k\sqrt{T}}e^{-\gamma t}\sin(k\pi t)\cos(x), \ k=(i-1)/T. \tag{7.6}$$

For $\left|\alpha\right|=2,\ \alpha_1=2,\ \alpha_i=0,\ i>1$ we get:

$$\frac{\partial}{\partial t}u(x,t)=\gamma\frac{\partial^2}{\partial x^2}u(x,t)+\sigma m_1(t)\frac{\partial}{\partial x}u_{|\alpha|=1,\alpha_1=1}(x,t),\quad u_\alpha(x,0)=0,$$

which results in:

$$u_\alpha(x,t)=\frac{-\sigma^2 t^2}{\sqrt{2}\,T}e^{-\gamma t}\sin(x). \tag{7.7}$$

For $\left|\alpha\right|=2,\ \alpha_1=0,\ \alpha_i=2,\ i>1$ we get:

$$\frac{\partial}{\partial t}u(x,t)=\gamma\frac{\partial^2}{\partial x^2}u(x,t)+\sqrt{2}\sigma m_i(t)\frac{\partial}{\partial x}u_{|\alpha|=1,\alpha_i=1}(x,t),\quad u_\alpha(x,0)=0,$$

which results in:

$$u_\alpha(x,t)=\frac{-\sqrt{2}\,\sigma^2}{T\,\pi^2 k^2}e^{-\gamma t}\sin^2(k\pi t)\sin(x). \tag{7.8}$$

For $\left|\alpha\right|=2,\ \alpha_1=1,\ \alpha_i=1,\ i>1$ we get:

$$\frac{\partial}{\partial t}u(x,t)=\gamma\frac{\partial^2}{\partial x^2}u(x,t)+\sigma m_1(t)\frac{\partial}{\partial x}u_{|\alpha|=1,\alpha_i=1}(x,t)+\sigma m_i(t)\frac{\partial}{\partial x}u_{|\alpha|=1,\alpha_1=1}(x,t),\quad u_\alpha(x,0)=0,$$

which results in:

$$u_\alpha(x,t)=\frac{-\sqrt{2}\,\sigma^2 t}{T\,\pi\ k}e^{-\gamma t}\sin(k\pi t)\ \ \sin(x). \tag{7.9}$$

For $\left|\alpha\right|=2,\ \alpha_i=1,\ \alpha_l=1,\ \ i,\ l>1$ we get:

$$\frac{\partial}{\partial t}u(x,t)=\gamma\frac{\partial^2}{\partial x^2}u(x,t)+\sigma m_i(t)\frac{\partial}{\partial x}u_{|\alpha|=1,\alpha j=1}(x,t)+\sigma mj(t)\frac{\partial}{\partial x}u_{|\alpha|=1,\alpha_i=1}(x,t),\ \ u_\alpha(x,0)=0;\ \ i,j>1,\ i<j,$$

which results in:

$$u_\alpha(x,t)=\frac{-2\sigma^2}{T\,\pi^2 k\,k_2}e^{-\gamma t}\sin(k\pi t)\sin(k_2\pi s)\sin(x),\quad k_2=j-1/T. \tag{7.10}$$

Then we have:

$$u_0=e^{-\gamma t}\sin(x),$$

$$u_{|\alpha|=1}=\frac{\sigma}{\sqrt{T}}e^{-\gamma t}\cos(x)\left[t+\frac{\sqrt{2}}{\pi k}\sin(k\pi t)\right],\ \ k=(i-1)/T,$$

$$u_{|\alpha|=2}=\frac{-\sigma^2}{T}e^{-\gamma t}\sin(x)\left[\frac{t^2}{\sqrt{2}}+\frac{\sqrt{2}\,t}{\pi\ k}\sin(k\pi t)+\frac{2}{\pi^2 k\,k_2}\sin(k\pi t)\sin(k_2\pi t)+\frac{\sqrt{2}}{\pi^2 k^2}\sin^2(k\pi t)\right],$$

$$,k_2=j-1/T.$$

The WHE differential equations (6.2), (6.3) and (6.4) can be solved using perturbation technique [7]. We can use the perturbation technique about advection coefficient σ combining with the Eigen function expansion (7.1) to calculate the first, second, third and fourth corrections.

Compare the coefficients of σ^0 :

$$\frac{\partial}{\partial t}u_0^0(x,t)-\mu\frac{\partial^2}{\partial x^2}u_0^0(x,t)=0,\quad u_0^0(x,0)=\sin(x),$$

to get $u_0^0(x,t)=e^{-\mu t}\sin(x),$ (7.11)

$$\frac{\partial}{\partial t}u_0^1(x,t;t_1)-\mu\frac{\partial^2}{\partial x^2}u_0^1(x,t;t_1)=0,\quad u_0^1(x,0;t_1)=0,$$

to get $u_0^1(x,t;t_1)=0,$ (7.12)

$$\frac{\partial}{\partial t}u_0^2(x,t;t_1,t_2)-\mu\frac{\partial^2}{\partial x^2}u_0^2(x,t;t_1,t_2)=0,\quad u_0^2(x,0;t_1,t_2)=0,$$

to get $u_0^2(x,t;t_1,t_2)=0.$ (7.13)

Compare the coefficients of σ^1 :

$$\frac{\partial}{\partial t}u_1^0(x,t)-\mu\frac{\partial^2}{\partial x^2}u_1^0(x,t)=\frac{\partial}{\partial x}u_0^1(x,t;t),\quad u_1^0(x,0)=0,$$

to get $u_1^0(x,t)=0,$ (7.14)

$$\frac{\partial}{\partial t}u_1^1(x,t;t_1)-\mu\frac{\partial^2}{\partial x^2}u_1^1(x,t;t_1)=\delta(t-t_1)\frac{\partial}{\partial x}u_0^0(x,t)+2\frac{\partial}{\partial x}u_0^2(x,t;t,t_1),\quad u_1^1(x,0;t_1)=0,$$

to get $u_1^1(x,t;t_1)=e^{-\mu t}\cos(x),$ (7.15)

$$\frac{\partial}{\partial t}u_1^2(x,t;t_1,t_2)-\mu\frac{\partial^2}{\partial x^2}u_1^2(x,t;t_1,t_2)=\frac{1}{2}\delta(t-t_2)\frac{\partial}{\partial x}u_0^1(x,t;t_1)+\frac{1}{2}\delta(t-t_1)\frac{\partial}{\partial x}u_0^1(x,t;t_2),$$
$$u_1^2(x,0;t_1,t_2)=0,$$

to get $u_1^2(x,t;t_1,t_2)=0.$ (7.16)

Compare the coefficients of σ^2 :

$$\frac{\partial}{\partial t}u_2^0(x,t)-\mu\frac{\partial^2}{\partial x^2}u_2^0(x,t)=\frac{1}{2}\frac{\partial^2}{\partial x^2}u_0^0(x,t)+\frac{\partial}{\partial x}u_1^1(x,t;t),\quad u_2^0(x,0)=0,$$

to get $u_2^0(x,t)=\frac{-3}{2}t\,e^{-\mu t}\sin(x),$ (7.17)

$$\frac{\partial}{\partial t}u_2^1(x,t;t_1)-\mu\frac{\partial^2}{\partial x^2}u_2^1(x,t;t_1)=\frac{1}{2}\frac{\partial^2}{\partial x^2}u_0^1(x,t;t_1)+\delta(t-t_1)\frac{\partial}{\partial x}u_1^0(x,t)+2\frac{\partial}{\partial x}u_1^2(x,t;t,t_1),$$
$$u_2^1(x,0;t_1)=0,$$

to get $u_2^1(x,t;t_1)=0,$ (7.18)

$$\frac{\partial}{\partial t}u_2^2(x,t;t_1,t_2)-\mu\frac{\partial^2}{\partial x^2}u_2^2(x,t;t_1,t_2)=\frac{1}{2}\frac{\partial^2}{\partial x^2}u_0^2(x,t;t_1,t_2)+\frac{1}{2}\delta(t-t_2)\frac{\partial}{\partial x}u_1^1(x,t;t_1)$$
$$+\frac{1}{2}\delta(t-t_1)\frac{\partial}{\partial x}u_1^1(x,t;t_2),\qquad u_2^2(x,0;t_1,t_2)=0,$$

to get $u_2^2(x,t;t_1,t_2)=-e^{-\mu t}\sin(x).$ (7.19)

Compare the coefficients of σ^3 :

$$\frac{\partial}{\partial t}u_3^0(x,t)-\mu\frac{\partial^2}{\partial x^2}u_3^0(x,t)=\frac{1}{2}\frac{\partial^2}{\partial x^2}u_1^0(x,t)+\frac{\partial}{\partial x}u_2^1(x,t;t),\quad u_3^0(x,0)=0,$$

to get $u_3^0(x,t)=0,$ (7.20)

$$\frac{\partial}{\partial t}u_3^1(x,t;t_1)-\mu\frac{\partial^2}{\partial x^2}u_3^1(x,t;t_1)=\frac{1}{2}\frac{\partial^2}{\partial x^2}u_1^1(x,t;t_1)+\delta(t-t_1)\frac{\partial}{\partial x}u_2^0(x,t)+2\frac{\partial}{\partial x}u_2^2(x,t;t,t_1),\quad u_3^1(x,0;t_1)=0,$$

to get $u_3^1(x,t;t_1)=-\frac{5}{2}t\,e^{-\mu t}\cos(x)-\frac{3}{2}t_1\,e^{-\mu t}\cos(x),$ (7.21)

$$\frac{\partial}{\partial t}u_3^2(x,t;t_1,t_2)-\mu\frac{\partial^2}{\partial x^2}u_3^2(x,t;t_1,t_2)=\frac{1}{2}\frac{\partial^2}{\partial x^2}u_1^2(x,t;t_1,t_2)+\frac{1}{2}\delta(t-t_2)\frac{\partial}{\partial x}u_2^1(x,t;t_1),$$
$$+\frac{1}{2}\delta(t-t_1)\frac{\partial}{\partial x}u_2^1(x,t;t_2),\qquad u_3^2(x,0;t_1,t_2)=0,$$

to get $u_3^2(x,t;t_1,t_2)=0.$ (7.22)

Compare the coefficients of σ^4 :

$$\frac{\partial}{\partial t}u_4^0(x,t)-\mu\frac{\partial^2}{\partial x^2}u_4^0(x,t)=\frac{1}{2}\frac{\partial^2}{\partial x^2}u_2^0(x,t)+\frac{\partial}{\partial x}u_3^1(x,t;t),\quad u_4^0(x,0)=0,$$

to get $u_4^0(x,t)=\frac{19}{8}t^2e^{-\mu t}\sin(x),$ (7.23)

$$\frac{\partial}{\partial t}u_4^1(x,t;t_1)-\mu\frac{\partial^2}{\partial x^2}u_4^1(x,t;t_1)=\frac{1}{2}\frac{\partial^2}{\partial x^2}u_2^1(x,t;t_1)+\delta(t-t_1)\frac{\partial}{\partial x}u_3^0(x,t)+2\frac{\partial}{\partial x}u_3^2(x,t;t,t_1),$$

to get $u_4^1(x,0;t_1)=0,\quad u_4^1(x,t;t_1)=0,$ (7.24)

$$\frac{\partial}{\partial t}u_4^2(x,t;t_1,t_2)-\mu\frac{\partial^2}{\partial x^2}u_4^2(x,t;t_1,t_2)=\frac{1}{2}\frac{\partial^2}{\partial x^2}u_2^2(x,t;t_1,t_2)+\frac{1}{2}\delta(t-t_2)\frac{\partial}{\partial x}u_3^1(x,t;t_1)$$
$$+\frac{1}{2}\delta(t-t_1)\frac{\partial}{\partial x}u_3^1(x,t;t_2),\qquad u_4^2(x,0;t_1,t_2)=0,$$

to get $u_4^2(x,t;t_1,t_2)=\frac{1}{2}te^{-\mu t}\sin(x)+2t_2e^{-\mu t}\sin(x)+2t_1e^{-\mu t}\sin(x).$ (7.25)

Then we have

$$u^{(0)}(x,t) = e^{-\mu t}\sin(x)(1-\frac{3}{2}\sigma^2 t+\frac{19}{8}\sigma^4 t^2),$$
$$u^{(1)}(x,t) = \sigma e^{-\mu t}\cos(x)\left(1-\frac{5}{2}\sigma^2(t+\frac{3}{5}t_1)\right), \quad (7.26)$$
$$u^{(2)}(x,t) = \sigma^2 e^{-\mu t}\sin(x)\left(-1+0.5\sigma^2\left(t+4t_1+4t_2\right)\right).$$

Then the mean and the variance are:

$$E[u(x,t)] = u^{(0)}(x,t) = e^{-\mu t}\sin(x)(1-\frac{3}{2}\sigma^2 t+\frac{19}{8}\sigma^4 t^2),$$
$$\mathrm{var}(u(x,t)) = \int_0^t [u^{(1)}(x,t;t_1)]^2 dt_1 + 2\int_0^t\int_0^t [u^{(2)}(x,t;t_1,t_2)]^2 dt_1 dt_2, \quad (7.27)$$
$$= \sigma^2 e^{-2\mu t}\cos^2(x)\left(t-\frac{13}{2}\sigma^2 t^2+\frac{43}{4}\sigma^4 t^3\right)+2\sigma^4 e^{-2\mu t}\sin^2(x)\left(\frac{t^2}{2}-\frac{5}{2}\sigma^2 t^3+\frac{83}{24}\sigma^4 t^4\right).$$

The WHEP technique gives the mean and the variance of the solution in the form of an infinite series in σ. Since σ represents the perturbation parameter in the WHEP technique, decreasing this parameter gives a good convergence.

VIII. NUMERICAL RESULTS

In order to examine the efficiency of the proposed methods, comparisons between the approximate solutions and the exact solution of the SADE are simulated through the following figures. We also introduce some discussion about the effect of the diffusion coefficient and advection coefficient.

We take $T=5$ and 50 Gaussian random variable. As shown in Figs (1-22), we can note, the approximate solution of the two methods and the exact solution are in satisfactory agreement with each other under some convergence conditions. The convergence of the WCE is enhanced by increasing the number of Gaussian random variables K. The convergence of the WHEP is enhanced by decreasing the perturbation parameter σ which represents the advection and also white noise coefficient.

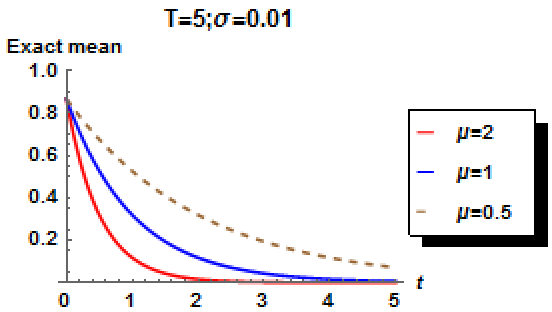

FIG 1. The exact mean at $x=\pi/3$ for different values of μ.

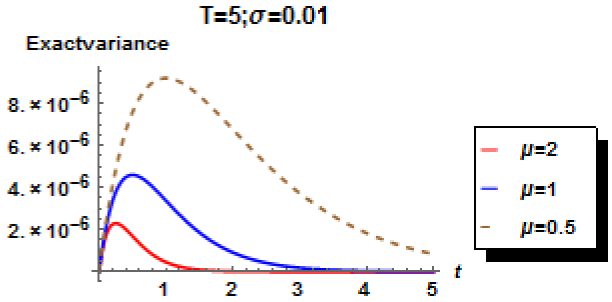

FIG 2. The exact variance at $x=\pi/3$ for different values of μ.

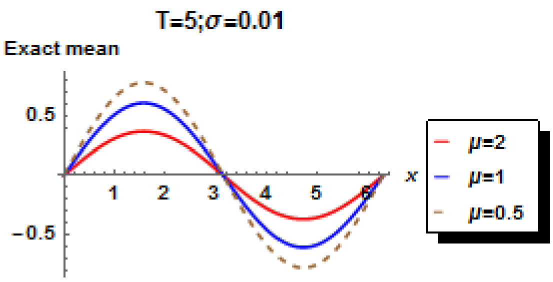

FIG 3. The exact mean at $t=0.5$ for different values of μ.

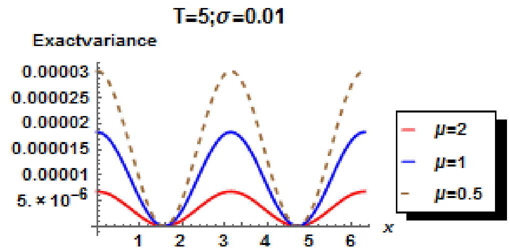

FIG 4. The exact variance at $t=0.5$ for different values of μ.

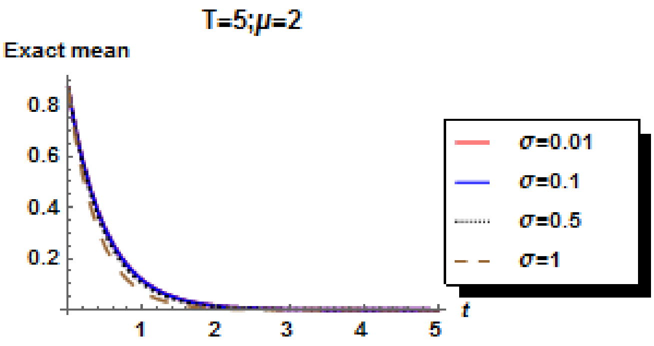

FIG 5. The exact mean at $x = \pi / 3$ for different values of σ .

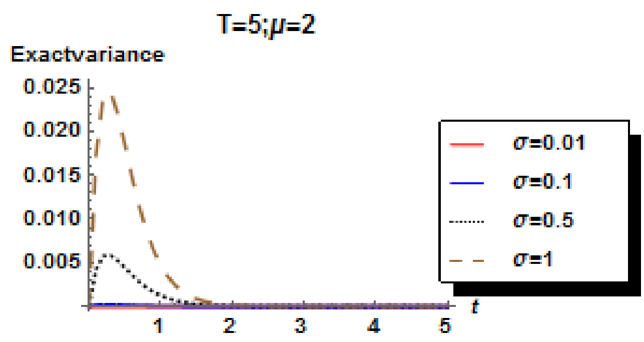

FIG 6. The exact variance at $x = \pi / 3$ for different values of σ .

First, we examine the effect of the diffusion coefficient μ and the advection coefficient σ on the exact solution with respect to the convergence rate (5.3) in Figs (1 - 6). Studying those figures, we can note that, for fixed point x, the decaying exponential function $e^{-\mu t}$ affects the solution. The mean solution and the variance solution decay faster with time at large values of diffusion coefficient μ . The larger the diffusion, the vanished faster the variance over time is. For fixed point t, the effect of the sinusoidal function appears on the behavior of the solution. As the diffusion coefficient μ increases, the variance decreases with time.

We investigate the effect of the advection coefficient σ on the solution of the mass transfer in Figs (5, 6). Examining those figures we found that, as the coefficient σ increases, the solution decreases and vanishes with the time. As the coefficient σ decreases, the variance also decreases. So choosing large value of μ and small value of σ is appropriate for good convergence of the approximate solution.

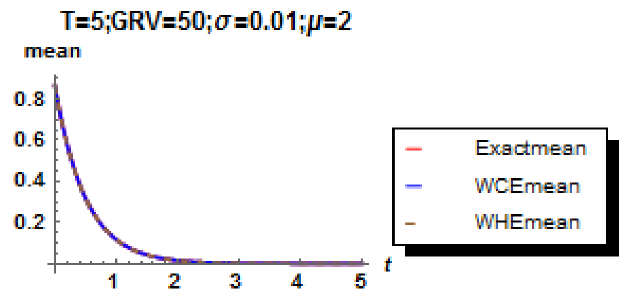

FIG 7. Second order means response for the exact, WCE and WHEP. Comparison between the three means at $x = \pi / 3$.

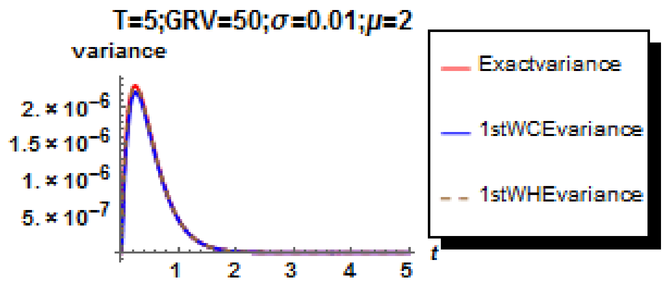

FIG 8. First order response variances for the exact, WCE and WHEP. Comparison between the three variances at $x = \pi / 3$.

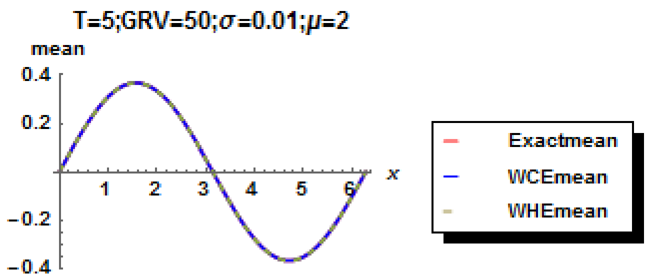

FIG 9. Second order means response for the exact, WCE and WHEP. Comparison between the three means at $t = 0.5$.

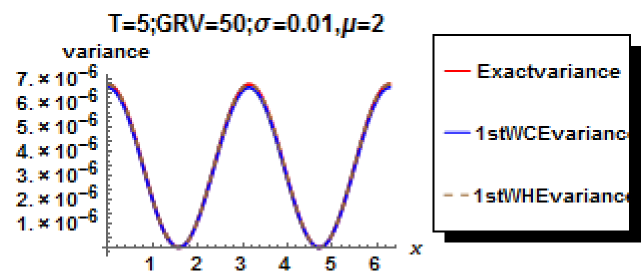

FIG 10. First order response variances for the exact, WCE and WHEP. Comparison between the three variances at $t = 0.5$.

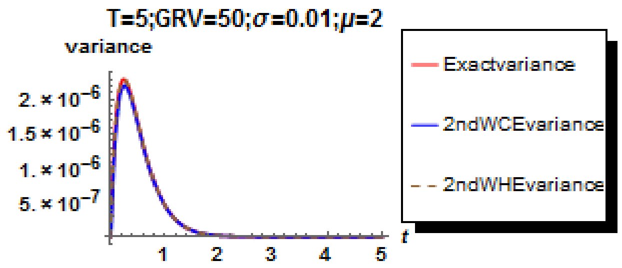

FIG 11. Second order response variances for the exact, WCE and WHEP. Comparison between the three variances at $x = \pi / 3$ **.**

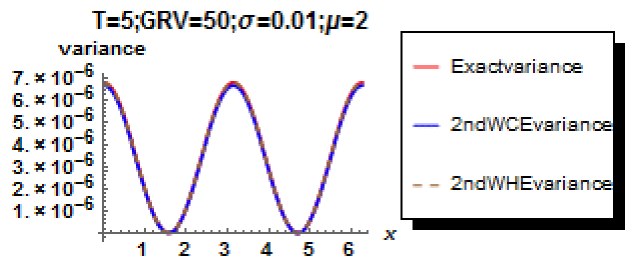

FIG 12. Second order response variances for the exact, WCE and WHEP. Comparison between the three variances at $t = 0.5$ **.**

The comparison between the exact solution and both approximate solutions of the WCE and WHEP techniques are shown in Figs (7-11). Examining those figures elucidate a satisfactory agreement between the exact solution and the approximate solutions either in mean or variance.

The effect of increasing the magnitude of the advection term σu_x is examined in Figs (13-16). We can note that increasing σ makes a deviation between the exact solution and the approximate solutions especially in the case of using the WHEP technique. This deviation is a result of using the WHEP technique which gives a mean solution and variance in the form of an infinite series in σ and t.

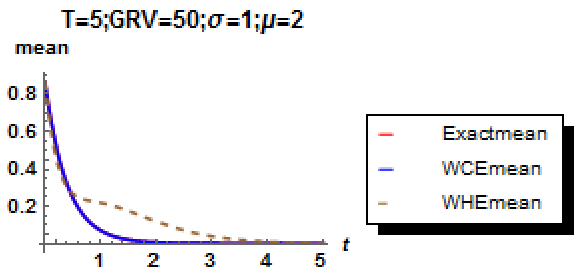

FIG 13. Second order means response for the exact, WCE and WHEP. Comparison between the three means at $x = \pi / 3, \sigma = 1$ **.**

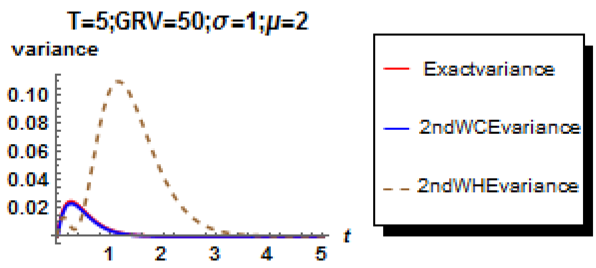

FIG 14. Second order response variances for the exact, WCE and WHEP. Comparison between the three variances at $x = \pi / 3, \sigma = 1$ **.**

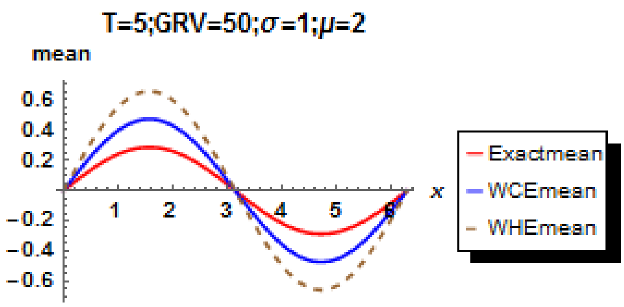

FIG 15. Second order means response for the exact, WCE and WHEP. Comparison between the three means at $t = 0.5, \sigma = 1$ **.**

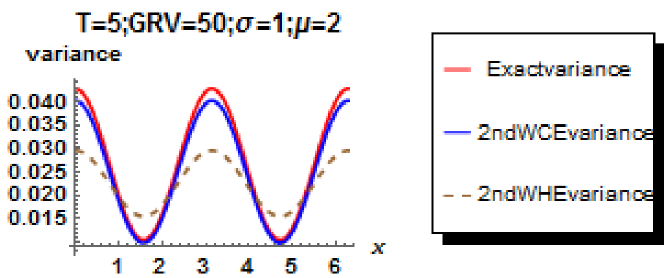

FIG 16. Second order response variances for the exact, WCE and WHEP. Comparison between the three variances at $t = 0.5, \sigma = 1$ **.**

From Figs (17, 18) we found that decreasing the number of GRVs leads to a deviation of the approximate solution of WCE. The larger the number of GRVs, the better convergence is.

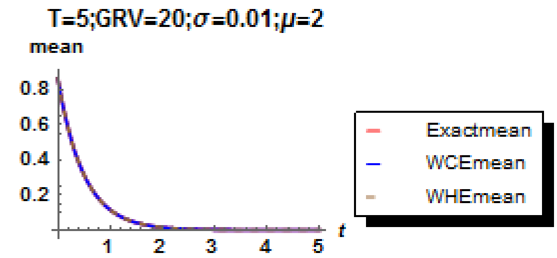

FIG 17. Second order means response for the exact, WCE and WHEP. Comparison between the three means at $x = \pi/3$ and $GRV = 20$.

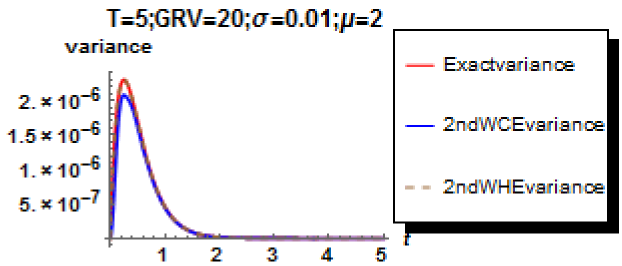

FIG 18. Second order response variances for the exact, WCE and WHEP. Comparison between the three variances at $x = \pi/3$ and $GRV = 20$.

Finally, we show the errors between the exact solution and the approximate solutions of both WCE and WHEP techniques. We investigate the effects of the coefficients of SADE on the error. In Fig 19 we found that, the error between the exact solution and the WCE approximate solution is decreased by increasing order solution of WCE. In Fig 20 we found that, as the diffusion coefficient μ increases, the error of second order approximate solution decreases.

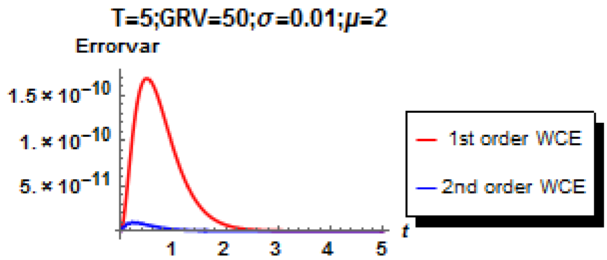

FIG 19. Error of variance response for 1st and 2nd order approximation of WCE. Comparison between the errors at $x = \pi/2$.

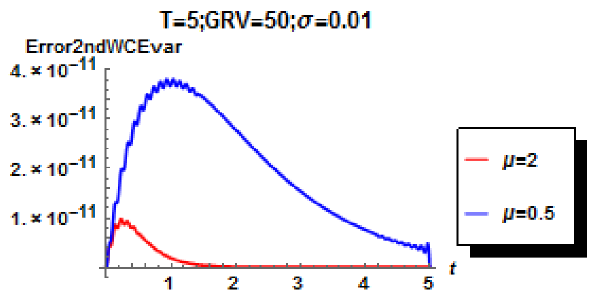

FIG 20. Error of second order approximation variance of WCE. Comparison between the errors for different value of μ at $x = \pi/2$.

In Fig (21-22) we found that, the error of WCE approximate solution is less than the error of the WHEP approximate solution. We can obtain the following conclusion: the WCE technique is better than the WHEP technique.

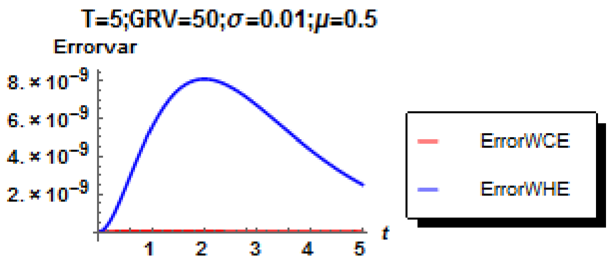

FIG 21. Second order response variance for the WCE and WHE. Comparison between the errors of variance at $x = \pi/2$ and $\mu = 0.5$.

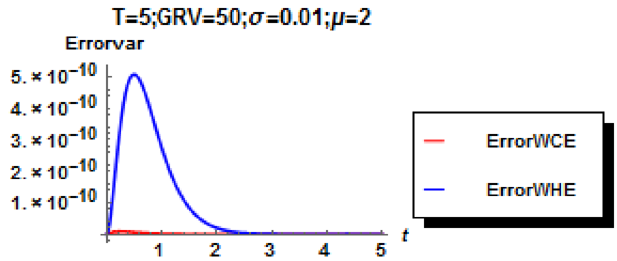

FIG 22. Second order response variance for the WCE and WHE. Comparison between the errors of variance at $x = \pi/2$ and $\mu = 2$.

IX. CONCLUSION

In this paper, the solution of the SADE with multiplicative white noise using WCE and WHEP techniques is introduced. The numerical results show that the WCE technique gives the closed form mean solution while the WHEP technique gives the mean solution in the form of an infinite series. Both techniques give the variance of the solution process in the form of infinite series.

The numerical results not only demonstrate the accuracy of the two techniques, but also show that the WCE technique is more efficient than the WHEP technique. The diffusion and advection coefficient and the intensity of Gaussian white noise play important roles in the SADE solution.

The convergence of the WCE depends on increasing the number of Gaussian random variables. The convergence of the WHEP technique depends on decreasing the perturbation parameter σ. The larger the diffusion coefficient, the more convergence the approximate solution is. The effect of advection coefficient is contrary to that of diffusion coefficient, i.e., the smaller the advection coefficient, the more convergence the approximate solution.

APPENDIX A

To prove the formula (7.1), let we have the general form of equation (2.2) with initial condition and a periodic boundary condition on x not for $-L \le x \le L$ but for $0 \le x \le 2\pi$,

$$u_t(x,t) = a\,u_{xx}(x,t) + F(x,t), \qquad u(x,0) = f(x),$$
$$u(0,t) = u(2\pi,t), \; \frac{du(0,t)}{dx} = \frac{du(2\pi,t)}{dx}, \tag{10.1}$$

for the homogenous case of equation (10.1), the solution will be in the form $u(x,t) = \sum_n T_n(t) Q_n(x)$, then we have

$$\frac{dT}{dt} = -a\lambda T, \quad \frac{d^2Q}{dx^2} + \lambda Q = 0, \tag{10.2}$$

there are three cases for λ. For $\lambda > 0$ the solution will be $Q(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$, the second case for $\lambda = 0$ the solution will be $Q(x) = A + Bx$ and the third case is $\lambda < 0$ and the solution will be $Q(x) = A\cosh(\sqrt{-\lambda}x) + B\sinh(\sqrt{-\lambda}x)$, after the application of boundary conditions for the three cases we will have the general Fourier series of a function with 2π period as

$$X(x) = \sum_{n=0}^{\infty} X_n^1 \cos(nx) + \sum_{n=1}^{\infty} X_n^2 \sin(nx) = \sum_{n,j} X_n^j Q_n^j(x), \tag{10.3}$$

with the Fourier coefficients

$$X_0 = \frac{1}{2\pi}\int_0^{2\pi} X(x)\,dx,$$
$$X_n^1 = \frac{1}{\pi}\int_0^{2\pi} X(x)\cos(nx)\,dx, \tag{10.4}$$
$$X_n^2 = \frac{1}{\pi}\int_0^{2\pi} X(x)\sin(nx)\,dx.$$

Let $u(x,t) = \sum_{n,j} u_n^j(t) Q_n^j(x)$ in equation (10.1) we get: $\sum_{n,j} \frac{d}{dt} u_n^j(t) Q_n^j(x) - a\sum_{n,j} u_n^j(t) Q_n^j(x)'' = \sum_{n,j} F_n^j(t) Q_n^j(x)$,

$$\sum_{n,j}\left[\frac{d}{dt}u_n^j(t) + a\lambda_n u_n^j(t) - F_n^j(t)\right] Q_n^j(x) = 0, \tag{10.5}$$

and for the initial condition,

$$f(x) = \sum_{n,j} C_n^j Q_n^j(x),$$
$$u(x,0) = \sum_{n,j} u_n^j(0) Q_n^j(x) = \sum_{n,j} C_n^j Q_n^j(x) \Rightarrow \sum_{n,j}\left[u_n^j(0) - C_n^j\right] Q_n^j(x) = 0, \tag{10.6}$$

$$\frac{d}{dt}u_n^j(t)+a\lambda_n u_n^j(t)-F_n^j(t)=0, \quad u_n^j(0)-C_n^j=0, \quad \lambda_n=n^2, \tag{10.7}$$

multipling both sides by $e^{-a\lambda_n t}$ and integrates then multiply by its orthognality then in the end we will get

$$u(x,t)=\sum_{n=0}^{\infty}\left[C_n^1 e^{-an^2t}+\int_0^t e^{-an^2(t-s)}F_n^1(s)\,ds\right]\cos(nx)+\sum_{n=1}^{\infty}\left[C_n^2 e^{-an^2t}+\int_0^t e^{-an^2(t-s)}F_n^2(s)\,ds\right]\sin(nx).$$

REFERENCES

[1] P. E. Kloeden, and E. Platen, Numerical Solution of Stochastic Differential Equations areds. Springer, Berlin, (1995).

[2] B. Oksendal, Stochastic Differential Equations: An Introduction with Application, Springer, Berlin, (1998).

[3] D. Nykamp and D. Tranchina. A population density method that facilitates large-scale modeling of neural networks: Extension to slow inhibitory synapses. *Neural Computation*, 13:511–546, 2001.

[4] N. Wiener. The homogeneous chaos. Amer. J. Math., 60:897–936, 1938.

[5] N. Wiener. Nonlinear Problems in Random Theory. MIT Press, Cambridge, 1958.

[6] W. C. Meecham and A. Siegel. Wiener-Hermite expansion in model turbulence at large Reynolds number. Phys. Fluids, 7:1178, 1964.

[7] M.A. El-Tawil, The Application of WHEP technique on stochastic partial differential equations, International Journal of Differential Equations and its Applications, Vol. 7, No.3, pp 325-337,2003.

[8] M.A. El-Beltagy and M.A El-Tawil, Toward a Solution of a Class of Non-Linear Stochastic perturbed PDEs Using Automated WHEP Algorithm, Applied Mathematical Modeling, 37 (2013) 7174–7192 7175.

[9] M.H. Abd-Elghany, M.A El-tawil, Wiener Hermite Expansion and Homotopy Method for Solving Stochastic DE: Wiener Hermite Expansion With Perturbation and Homotopy Techniques, Noor Publishing (2016), 188.pp.

[10] M. El-Beltagy and A. Al-Johani, "Higher-Order WHEP Solutions of Quadratic Nonlinear Stochastic Oscillatory Equation," Engineering, Vol. 5 No. 5A,(2013), pp. 57-69. doi: 10.4236/eng.2013.55A00.

[11] M.A. El-Tawil, and A. Fareed, "Solution of Sto-chastic Cubic and Quintic Nonlinear Diffusion Equation Using WHEP, Pickard and HPM Meth-ods," Open Journal of Discrete Mathematics, Vol. 1 No. 1, 2011, pp. 6-21.
doi:10.4236/ojdm.2011.11002.

[12] H. Cameron and W. T. Martin, "The Orthogonal Development of Non-linear Functionals in Series of Fourier- Hermite Functionals, Ann. Math., Vol. 48 (1947) pp. 385–392.

[13] H. Holden, B. Øksendal, J. Uboe, and T. Zhang. Stochastic Partial Differential Equations. A Modeling, White Noise Functional Approach. Birkh¨auser, Boston, 1996.

[14] R. Mikulevicius and B. Rozovskii. Linear parabolic stochastic PDEs and Wiener chaos. SIAM J. Math. Anal., 29:452–480, 1998.

[15] R. Mikulevicius and B. Rozovskii. Stochastic Navier-Stokes equations for turbulence flow. SIAM J. Math. Anal., 35:1250–1310, 2004.

[16] R. Mikulevicius and B. L. Rozovskii. Stochastic Navier-Stokes equations: Propagation of chaos and statistical moments. In J. L.Menaldi, E. Rofmann, and A. Sulem, editors, Optimal Control and Partial Differential Equations, pages 258–267, Amsterdam, 2001. IOS Press. In honor of Professor Alain Bensoussan's 60th Birthday.

[17] S. Lototsky, R. Mikulevicius, and B. Rozovskii. Nonlinear filtering revisited: a spectral approach. SIAM J. Control Optim., 35:435–461, 1997.

[18] S. V. Lototsky. Nonlinear filtering of diffusion processes in correlated noise: analysis by separation of variables. Appl. Math. Optim., 47(2):167–194, 2003.

[19] R. G. Ghanem. Stochastic finite element with multiple random non-Gaussian properties. J. Engrg. Mech., 125(1):26–40, 1998.

[20] S. Sakamoto and R. Ghanem. Polynomial chaos decomposition for the simulation of non-Gaussian non-stationary stochastic processes. J. Eng. Mech., 128:190–201, 2002.

[21] I. Babuska, R. Tempone, and G. E. Zouraris. Solving elliptic boundary value problems with uncertain coefficients by the finite element method: the stochastic formulation. Comput. Methods Appl. Mech. Engrg., 194:1251–1294, 2005.

[22] P. Frauenfelder, C. Schwab, and R. A. Todor. Finite elements for elliptic problems with stochastic coefficients. Comput. Method Appl. Mech. Engrg., 194:205–228, 2005.

[23] H. G. Matthies and A. Keese. Galerkin methods for linear and nonlinear elliptic stochastic partial differential equations. Comput. Methods Appl. Mech. Engrg., 194:1295 -1331, 2005.

[24] D. Xiu and G. E. Karnidakis. The Wiener-Askey polynomial chaos for stochastic differential equations. SIAM J. Sci. Comput., 24:619–644, 2002.

[25] D. Zhang and Z. Lu. An efficient, high-order perburbation approach for flow in random porous media via Karhunen-Loeve and polynomial expansion. J. Comput. Phy., 194:137–167, 2003.

[26] Lue W., "Wiener Chaos Expansion and Numerical Solutions of Stochastic Partial Differential Equations", PhD Thesis, California Institute of Technology, Pasadena, California, (2006).

[27] Z. Zhang, B. Rozovskii, M. V. Tretyakov, and G. E. Karniadakis,A Multistage Wiener Chaos Expansion Method for Stochastic Advection-Diffusion-Reaction Equations, *SIAM J. Sci. Comput.*, 34(2), (2012) A914–A936.

[28] T. Imamura, W. Meecham, A. Siegel, Symbolic Calculus of the Wiener Process and Wiener-Hermite Functional, J. Math. Phys. Vol. 6, No. 5, (1965), pp. 695–706.

[29] M. El Tawil, M. Elbeltagy, B.S. El-desouky and M. Hamed, Solution of Nonlinear Stochastic Langevin's Equation Using WHEP, Pickard and HPM Methods, Applied Mathematics, 2014, 5, 398-412.

[30] David H. Bailey, Paul N. Swarztrauber, "A fast method for the numerical evaluation of continuous Fourier and Laplace transforms", SIAM Journal on Scientific Computing , 15 (5): 1105–1110 , (1994), doi:10.1137/0915067.