Dynamic analyses of a flat plate and a beam subjected to a moving load

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Abstract— *The objective of this paper is to investigate the dynamic characteristics of a flat plate and a beam subjected to a moving load with the effects of inertia force, Coriolis force and centrifugal force considered. To this end, the theory of moving mass element for plate and that for beam are presented, where the property matrices of the last elements are derived based on the superposition principle and the definition of shape functions. It is found that the order of the property matrices of the moving mass element for plate is* 24×24*, while that for beam is* 12×12*. Combination of the property matrices of the moving mass element and the overall property matrices of the plate (or beam) itself gives the overall property matrices of the entire structural system. Because the property matrices of the moving mass element have close relationship with the instantaneous position of the moving load, they are time-variant and so are the overall property matrices of the entire structural system. For validation, the vibration characteristics of the rectangular plate due to a moving load are compared with those of the beam, with its sizes being the same as those of the plate, due to the same loading conditions and satisfactory agreement is achieved. Some factors closely relating to the title problem, such as the moving-load speed, acceleration, inertia force, Coriolis force and centrifugal force, are investigated. Numerical results reveal that all the above-mentioned parameters affecting the dynamic responses of the plate to some degree.*

Keywords— **Dynamic responses, Plate, Beam, Moving mass element, Moving load.**

I. INTRODUCTION

Flexible structures undergoing external loadings are important research topics. For the structures with slender shapes, the theories of beams are usually adopted in the analyses. However, if the structures have plate-like (rather than beam-like) shapes, the theories of beams are no longer available for accurately estimating the dynamic behaviour of the structures. For this reason, many researchers have studied the vibration characteristics of structures by means of the plate theories [1-14]. For instance, Manoach [1] has studied the dynamic behaviour of elastoplastic thick circular plates due to different types of pulses by using the Mindlin plate theory. Wu, Lee and Lai [2] have performed the forced vibration analyses of a flat plat under various moving loads with finite element method incorporated with the Newmark direct integration method. Gbadeyan and Oni [3], Frýba [4], Lin [5], Shadnam, Rofooei and Mehri [6] and Renard and Taazount [7] have investigated the dynamic behaviour of beams and plates subjected to moving forces and moving masses. Marchesiello et al. [8] and Chatterjee and Datta [9] have, respectively, studied the dynamic behaviour of multi-span bridges and arch bridges under moving vehicle loads, where the bridges are modelled as plates. Takabatake [10] has presented a simplified analytical method for calculating the dynamic responses of a rectangular plate with stepped thickness and subjected to moving loads. Rossi, Gutierrez and Laura [11] have studied the forced vibration responses of a rectangular plate undergoing a stationary distributed harmonic loading. Shadnam, Mofid and Akin [12] have formulated the forced vibration problem of a rectangular plate due to a single force (or mass) moving along an arbitrary trajectory by means of the analytical and numerical approaches. Wu [13] and Kononov and Borst [14] have researched the vibration characteristics of a plate due to forces moving along a circular path, respectively, using the finite element method and the analytical approach. From the literature listed above, it is found that the effects of inertia force, Coriolis force and centrifugal force, induced by the moving load, cannot easily take into account the dynamic responses of the plate. To solve this problem, the theory of moving mass element is presented in this paper.

Firstly, under the assumption that the moving load is regarded as a concentrated mass, the property matrices of the moving mass element for plate are derived based on the superposition principle and the definition of shape functions. Because the property matrices of the moving mass element have something to do with the instantaneous position of the moving load, the last matrices vary with time. Adding the property matrices of the moving mass element to the overall property matrices of the plate itself yields the time-variant overall property matrices of the entire structural system. For validation, the property matrices of the moving mass element for beam are also derived and the vibration characteristics of the rectangular plate due to a moving load are compared with those of the beam, with its sizes being the same as those of the plate, due to the same loading conditions and satisfactory agreement is achieved. Some pertinent factors closely relating to the title problem, such as the moving-load speed, acceleration, inertia force, Coriolis force and centrifugal force, are studied. Numerical results reveal that all the above-mentioned parameters have significant influences on the dynamic responses of the plate.

II. PROPERTY MATRICES OF MOVING MASS ELEMENT FOR PLATE

Figure 1 shows the s_p^{th} plate element subjected to a moving concentrated mass m_c at the instant of time *t*. Since the concentrated mass m_c is moving along a vibrating path on the plate, the vertical ($\bar{z}$) velocity and acceleration of the moving mass are respectively given by

$$\dot{w}_{\bar{z}}(x,y,t) = \frac{\partial w_{\bar{z}}}{\partial x}\dot{x} + \frac{\partial w_{\bar{z}}}{\partial y}\dot{y} + \dot{w}_{\bar{z}} \tag{1a}$$

$$\ddot{w}_{\bar{z}}(x,y,t) = \frac{\partial^2 w_{\bar{z}}}{\partial x^2}\dot{x}^2 + 2\frac{\partial^2 w_{\bar{z}}}{\partial x \partial y}\dot{x}\dot{y} + 2\frac{\partial^2 w_{\bar{z}}}{\partial x \partial t}\dot{x} + 2\frac{\partial^2 w_{\bar{z}}}{\partial y \partial t}\dot{y} + \frac{\partial^2 w_{\bar{z}}}{\partial y^2}\dot{y}^2 + \frac{\partial w_{\bar{z}}}{\partial x}\ddot{x} + \frac{\partial w_{\bar{z}}}{\partial y}\ddot{y} + \ddot{w}_{\bar{z}} \tag{1b}$$

where $w_{\bar{z}} \equiv w_{\bar{z}}(x,y,t)$, $\dot{w}_{\bar{z}}(x,y,t)$ and $\ddot{w}_{\bar{z}}(x,y,t)$, respectively, represent the vertical ($\bar{z}$) displacement, velocity and acceleration of the moving mass m_c at position (x , y) and time t ; $\dot{x}$ and $\dot{y}$, respectively, represent the velocities of the moving mass m_c in the $\bar{x}$ and $\bar{y}$ directions (cf. Figure 2); while $\ddot{x}$ and $\ddot{y}$, respectively, represent the accelerations of that in the $\bar{x}$ and $\bar{y}$ directions. It is noted that the mass is assumed to be in close contact with the plate at all time.

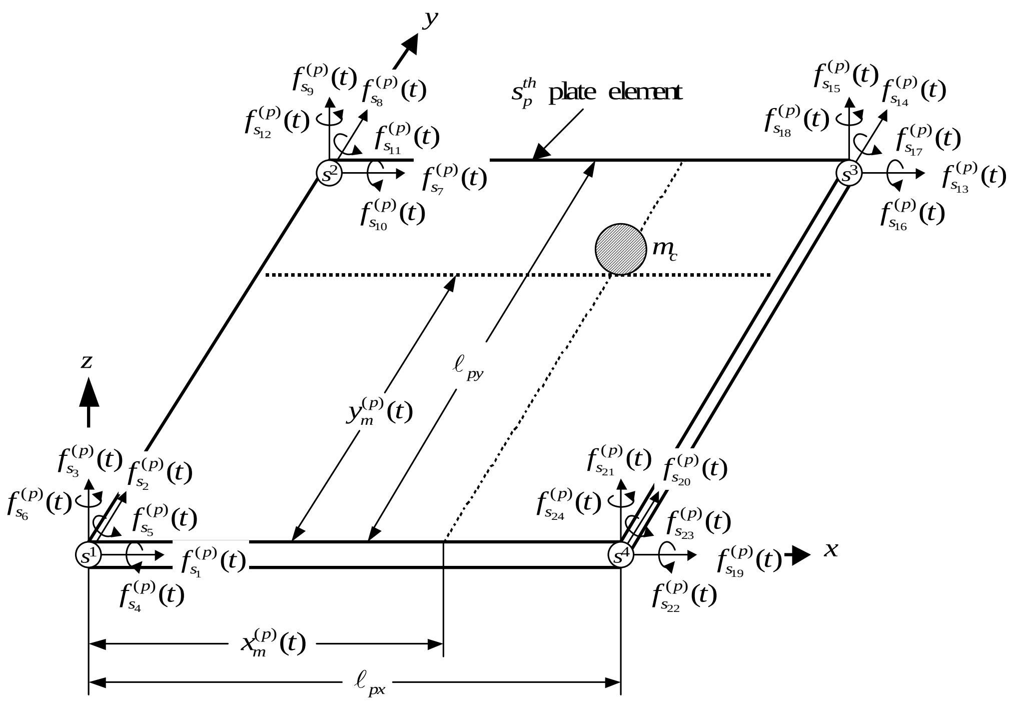


FIGURE 1 THE s_p^{th} PLATE ELEMENT SUBJECTED TO A MOVING LOAD AT THE INSTANT OF TIME t .

For convenience, Equation (1b) is re-written as

$$\ddot{w}_{\bar{z}}(x,y,t) = w_{\bar{z}}^{xx}V_{mx}^2 + 2w_{\bar{z}}^{xy}V_{mx}V_{my} + 2\dot{w}_{\bar{z}}^{x}V_{mx} + 2\dot{w}_{\bar{z}}^{y}V_{my} + w_{\bar{z}}^{yy}V_{my}^2 + w_{\bar{z}}^{x}\dot{V}_{mx} + w_{\bar{z}}^{y}\dot{V}_{my} + \ddot{w}_{\bar{z}} \tag{2}$$

where the superscripts x , y and dot, respectively, represent the derivatives with respect to x , y and time t ; while $V_{mx} \equiv \dot{x}$, $V_{my} \equiv \dot{y}$, $\dot{V}_{mx} \equiv \ddot{x}$ and $\dot{V}_{my} \equiv \ddot{y}$.

If the plate is vibrating, then $\bar{x}$, $\bar{y}$ and $\bar{z}$ force components of the contact point, between the moving mass and the plate element, induced by the vibration and curvature of the plate element are, respectively, given by

$$f_{\bar{x}}(x,y,t) = m_c \ddot{w}_{\bar{x}} \delta(x - V_{mx}t)\delta(y - V_{my}t) \tag{3a}$$

$$f_{\bar{y}}(x,y,t) = m_c \ddot{w}_{\bar{y}} \delta(x - V_{mx}t)\delta(y - V_{my}t) \tag{3b}$$

$$\begin{aligned} f_{\bar{z}}(x,y,t) = m_c [& w_{\bar{z}}^{xx} V_{mx}^2 + 2w_{\bar{z}}^{xy} V_{mx} V_{my} + 2\dot{w}_{\bar{z}}^{x} V_{mx} \\ & + 2\dot{w}_{\bar{z}}^{y} V_{my} + w_{\bar{z}}^{yy} V_{my}^2 + w_{\bar{z}}^{x} \dot{V}_{mx} + w_{\bar{z}}^{y} \dot{V}_{my} + \ddot{w}_{\bar{z}}]\delta(x - V_{mx}t)\delta(y - V_{my}t) \end{aligned} \tag{3c}$$

where $\delta(x - V_{mx}t)$ and $\delta(y - V_{my}t)$ represent the Dirac delta function; while $\ddot{w}_{\bar{x}} \equiv \ddot{w}_{\bar{x}}(x,y,t)$ and $\ddot{w}_{\bar{y}} \equiv \ddot{w}_{\bar{y}}(x,y,t)$, respectively, represent the $\bar{x}$ and $\bar{y}$ accelerations of the plate element at position (x , y) and time t .

In such a case, the equivalent nodal forces, $f_{s_k}^{(p)}$ (k =1 to 24), of the s_p^{th} plate element are given by [13]

$$f_{s_k}^{(p)} = \Phi_k^{(p)} f_{\bar{x}}(x,y,t) \qquad (k = 1,7,13,19) \tag{4a}$$

$$f_{s_k}^{(p)} = \Phi_k^{(p)} f_{\bar{y}}(x,y,t) \qquad (k = 2,8,14,20) \tag{4b}$$

$$f_{s_k}^{(p)} = \Phi_k^{(p)} f_{\bar{z}}(x,y,t) \qquad (k = 3,4,5,9,10,11,15,16,17,21,22,23) \tag{4c}$$

$$f_{s_k}^{(p)} = 0 \qquad (k = 6,12,18,24) \tag{4d}$$

where $\Phi_k^{(p)}$ (k =1 to 24) are shape functions with the non-zero ones given by [13]

$$\begin{aligned}
&\Phi_1^{(p)} = \Phi_2^{(p)} = (1-\varsigma_p)(1-\eta_p)\,,\ \Phi_3^{(p)} = (1+2\varsigma_p)(1-\varsigma_p)^2(1+2\eta_p)(1-\eta_p)^2 \\
&\Phi_4^{(p)} = (1+2\varsigma_p)(1-\varsigma_p)^2 \eta_p (1-\eta_p)^2 \ell_{py}\,,\ \Phi_5^{(p)} = -(1-\varsigma_p)^2 \varsigma_p (1+2\eta_p)(1-\eta_p)^2 \ell_{px} \\
&\Phi_7^{(p)} = \Phi_8^{(p)} = (1-\varsigma_p)\eta_p\,,\ \Phi_9^{(p)} = (1+2\varsigma_p)(1-\varsigma_p)^2(3-2\eta_p)\eta_p^2 \\
&\Phi_{10}^{(p)} = -(1+2\varsigma_p)(1-\varsigma_p)^2(1-\eta_p)\eta_p^2 \ell_{py}\,,\ \Phi_{11}^{(p)} = -\varsigma_p(1-\varsigma_p)^2(3-2\eta_p)\eta_p^2 \ell_{px} \\
&\Phi_{13}^{(p)} = \Phi_{14}^{(p)} = \varsigma_p \eta_p\,,\ \Phi_{15}^{(p)} = (3-2\varsigma_p)\varsigma_p^2(3-2\eta_p)\eta_p^2 \\
&\Phi_{16}^{(p)} = -(3-2\varsigma_p)\varsigma_p^2(1-\eta_p)\eta_p^2 \ell_{py}\,,\ \Phi_{17}^{(p)} = (1-\varsigma_p)\varsigma_p^2(3-2\eta_p)\eta_p^2 \ell_{px} \\
&\Phi_{19}^{(p)} = \Phi_{20}^{(p)} = \varsigma_p(1-\eta_p)\,,\ \Phi_{21}^{(p)} = (3-2\varsigma_p)\varsigma_p^2(1+2\eta_p)(1-\eta_p)^2 \\
&\Phi_{22}^{(p)} = (3-2\varsigma_p)\varsigma_p^2 \eta_p (1-\eta_p)^2 \ell_{py}\,,\ \Phi_{23}^{(p)} = (1-\varsigma_p)\varsigma_p^2(1+2\eta_p)(1-\eta_p)^2 \ell_{px}
\end{aligned} \tag{5a}$$

$$\varsigma_p = x_m^{(p)}(t) \big/ \ell_{px}\ ,\ \eta_p = y_m^{(p)}(t) \big/ \ell_{py} \tag{5b}$$

where ℓ_{px} and ℓ_{py} are respectively the length and width of the rectangular plate element (see Figure 1), whereas $x_m^{(p)}(t)$ and $y_m^{(p)}(t)$ are respectively the local *x* and *y* positions of the concentrated mass m_c at time *t*.

Based on the superposition principle and the definition of shape functions, the local x, y and z displacements of contact point, w_x, w_y and w_z, can be obtained from

$$w_x \equiv w_{\bar{x}} = \Phi_1^{(p)} u_{s_1}^{(p)} + \Phi_7^{(p)} u_{s_7}^{(p)} + \Phi_{13}^{(p)} u_{s_{13}}^{(p)} + \Phi_{19}^{(p)} u_{s_{19}}^{(p)} \quad (6a)$$

$$w_y \equiv w_{\bar{y}} = \Phi_2^{(p)} u_{s_2}^{(p)} + \Phi_8^{(p)} u_{s_8}^{(p)} + \Phi_{14}^{(p)} u_{s_{14}}^{(p)} + \Phi_{20}^{(p)} u_{s_{20}}^{(p)} \quad (6b)$$

$$\begin{aligned} w_z \equiv w_{\bar{z}} = &\Phi_3^{(p)} u_{s_3}^{(p)} + \Phi_4^{(p)} u_{s_4}^{(p)} + \Phi_5^{(p)} u_{s_5}^{(p)} + \Phi_9^{(p)} u_{s_9}^{(p)} + \Phi_{10}^{(p)} u_{s_{10}}^{(p)} + \Phi_{11}^{(p)} u_{s_{11}}^{(p)} \\ &+ \Phi_{15}^{(p)} u_{s_{15}}^{(p)} + \Phi_{16}^{(p)} u_{s_{16}}^{(p)} + \Phi_{17}^{(p)} u_{s_{17}}^{(p)} + \Phi_{21}^{(p)} u_{s_{21}}^{(p)} + \Phi_{22}^{(p)} u_{s_{22}}^{(p)} + \Phi_{23}^{(p)} u_{s_{23}}^{(p)} \end{aligned} \quad (6c)$$

where $u_{s_i}^{(p)}$ (i =1 to 5, 7 to 11, 13 to 17 and 19 to 23) are the nodal displacements of the nodes of the plate element on which the moving concentrated mass m_c applies.

Introducing Equations (3) and (6) into Equation (4), and writing the resulting expressions in matrix form yields

$$\{f^{(p)}\} = [m^{(p)}]\{\ddot{u}^{(p)}\} + [c^{(p)}]\{\dot{u}^{(p)}\} + [k^{(p)}]\{u^{(p)}\} \quad (7a)$$

$$\{f^{(p)}\} = [f_{s_1}^{(p)} \quad f_{s_2}^{(p)} \quad \cdots \quad f_{s_{23}}^{(p)} \quad f_{s_{24}}^{(p)}]^T \quad (7b)$$

$$\{\ddot{u}^{(p)}\} = [\ddot{u}_{s_1}^{(p)} \quad \ddot{u}_{s_2}^{(p)} \quad \cdots \quad \ddot{u}_{s_{23}}^{(p)} \quad \ddot{u}_{s_{24}}^{(p)}]^T \quad (7c)$$

$$\{\dot{u}^{(p)}\} = [\dot{u}_{s_1}^{(p)} \quad \dot{u}_{s_2}^{(p)} \quad \cdots \quad \dot{u}_{s_{23}}^{(p)} \quad \dot{u}_{s_{24}}^{(p)}]^T \quad (7d)$$

$$\{u^{(p)}\} = [u_{s_1}^{(p)} \quad u_{s_2}^{(p)} \quad \cdots \quad u_{s_{23}}^{(p)} \quad u_{s_{24}}^{(p)}]^T \quad (7e)$$

$$[m^{(p)}] = m_c [m^{(1)}]_{24\times24} \quad (8a)$$

$$[c^{(p)}] = 2m_c V_{mx} [c^{(1)}]_{24\times24} + 2m_c V_{my} [c^{(2)}]_{24\times24} \quad (8b)$$

$$\begin{aligned} [k^{(p)}] = &m_c V_{mx}^2 [k^{(1)}]_{24\times24} + 2m_c V_{mx} V_{my} [k^{(2)}]_{24\times24} \\ &+ m_c V_{my}^2 [k^{(3)}]_{24\times24} + m_c \dot{V}_{mx} [k^{(4)}]_{24\times24} + m_c \dot{V}_{my} [k^{(5)}]_{24\times24} \end{aligned} \quad (8c)$$

In Equation (8), all the coefficients of $[m^{(1)}]_{24\times24}$, $[c^{(1)}]_{24\times24}$, $[c^{(2)}]_{24\times24}$, $[k^{(1)}]_{24\times24}$, $[k^{(2)}]_{24\times24}$, $[k^{(3)}]_{24\times24}$, $[k^{(4)}]_{24\times24}$ and $[k^{(5)}]_{24\times24}$ are equal to zero except

$$m_{ij}^{(1)} = \Phi_i^{(p)} \Phi_j^{(p)} \quad (i, j = 1,7,13,19) \quad (9a)$$

$$m_{ij}^{(1)} = \Phi_i^{(p)} \Phi_j^{(p)} \quad (i, j = 2,8,14,20) \quad (9b)$$

$$m_{ij}^{(1)} = \Phi_i^{(p)} \Phi_j^{(p)} \quad (i, j = 3,4,5,9,10,11,15,16,17,21,22,23) \quad (9c)$$

$$c_{ij}^{(1)} = k_{ij}^{(4)} = \Phi_i^{(p)} \Phi_j^{(p)x} \quad (i, j = 3,4,5,9,10,11,15,16,17,21,22,23) \quad (9d)$$

$$c_{ij}^{(2)} = k_{ij}^{(5)} = \Phi_i^{(p)} \Phi_j^{(p)y} \quad (i, j = 3,4,5,9,10,11,15,16,17,21,22,23) \quad (9e)$$

$$k_{ij}^{(1)} = \Phi_i^{(p)} \Phi_j^{(p)xx} \quad (i, j = 3,4,5,9,10,11,15,16,17,21,22,23) \quad (9f)$$

$$k_{ij}^{(2)} = \Phi_i^{(p)} \Phi_j^{(p)xy} \quad (i, j = 3,4,5,9,10,11,15,16,17,21,22,23) \quad (9g)$$

$$k_{ij}^{(3)} = \Phi_i^{(p)} \Phi_j^{(p)yy} \quad (i, j = 3,4,5,9,10,11,15,16,17,21,22,23) \quad (9h)$$

It is noted that the superscripts x and y represent the derivatives with respect to x and y, respectively.

In Equation (8), $[m^{(p)}]$, $[c^{(p)}]$ and $[k^{(p)}]$ are the mass, damping and stiffness matrices of the presented *moving mass element for plate*. Since the last matrices have something to do with the shape functions of the plate element on which the moving mass applies (see Equations (5) and (9)), their properties change from time to time if the position of concentrated mass m_c on the plate is time-variant.

III. DYNAMIC RESPONSES OF A FLAT PLATE DUE TO A MOVING LOAD

For a multiple degree of freedom damped structural system, its equations of motion is given by [15]

$$[\overline{M}]\{\ddot{\overline{q}}(t)\}+[\overline{C}]\{\dot{\overline{q}}(t)\}+[\overline{K}]\{\overline{q}(t)\}=\{\overline{F}(t)\} \quad (10)$$

where $[\overline{M}]$, $[\overline{C}]$ and $[\overline{K}]$ are, respectively, the overall mass, damping and stiffness matrices; $\{\ddot{\overline{q}}(t)\}$, $\{\dot{\overline{q}}(t)\}$ and $\{\overline{q}(t)\}$ are, respectively, the acceleration, velocity and displacement vectors, whereas $\{\overline{F}(t)\}$ is the overall external force vector at any time *t*. For a plate subjected to a moving load, the overall property matrices, $[\overline{M}]$, $[\overline{C}]$ and $[\overline{K}]$, and the external force vector $\{\overline{F}(t)\}$ are determined using the expressions introduced in the subsequent subsections.

3.1 Overall property matrices of the entire structural system

To take the effects of inertial force and centrifugal force of the moving load into account, the overall mass matrix, $[\overline{M}]$, and stiffness matrix, $[\overline{K}]$, of the entire structural system must be determined by adding the mass and stiffness matrices, $[m^{(p)}]$ and $[k^{(p)}]$, of the moving mass element to the overall ones of the plate itself, i.e.,

$$[\overline{M}]=[M_p]+[m^{(p)}]_{24\times24} \quad (11a)$$

$$[\overline{K}]=[K_p]+[k^{(p)}]_{24\times24} \quad (11b)$$

and imposing the prescribed boundary conditions. In which, $[M_p]$ and $[K_p]$ are, respectively, the overall mass and stiffness matrices of the plate itself and may be obtained by assembling all its element mass and stiffness matrices [16].

In Equation (11), all the coefficients of the mass and stiffness matrices, $[\overline{M}]$ and $[\overline{K}]$, are exactly the same as the corresponding ones of $[M_p]$ and $[K_p]$, i.e.,

$$\overline{M}_{ij}=M_{p,ij} \quad (12a)$$

$$\overline{K}_{ij}=K_{p,ij} \quad (i,j=1 \text{ to } n) \quad (12b)$$

except

$$\overline{M}_{D_iD_j}=M_{p,D_iD_j}+m_{ij}^{(p)} \quad (13a)$$

$$\overline{K}_{D_iD_j}=K_{p,D_iD_j}+k_{ij}^{(p)} \quad (i,j=1 \text{ to } 24) \quad (13b)$$

In Equations (12)-(13), *n* and D_i ($i=1$ to 24) are, respectively, the total degrees of freedom of the entire structural system and the numberings for the 24 degrees of freedom for the four nodes of the D^{th} plate element on which the moving load applies at time *t*.

Due to the fact that the elementary damping matrix of the structural system is difficult to find from the existing literature, the overall damping matrix $[C_p]$ of the plate itself is assumed to be proportional and determined by using the Rayleigh damping theory [15].

$$[C_p] = \alpha[\overline{M}] + \beta[\overline{K}] \tag{14a}$$

$$\alpha = \frac{2\omega_i\omega_j(\xi_i\omega_j - \xi_j\omega_i)}{\omega_j^2 - \omega_i^2} \tag{14b}$$

$$\beta = \frac{2(\xi_j\omega_j - \xi_i\omega_i)}{\omega_j^2 - \omega_i^2} \tag{14c}$$

where ξ_i and ξ_j represent the damping ratios corresponding to any two natural frequencies of the structural system, ω_i and ω_j, respectively.

In such a case, the overall damping matrix $[\overline{C}]$ of the entire structural system can be obtained from

$$[\overline{C}] = [C_p] + [c^{(p)}]_{24\times24} \tag{15a}$$

$$\overline{C}_{ij} = C_{p,ij} \quad (i, j = 1 \text{ to } n) \tag{15b}$$

except

$$\overline{C}_{D_iD_j} = C_{p,D_iD_j} + c_{ij}^{(p)} \qquad (i, j = 1 \text{ to } 24) \tag{15c}$$

where the meanings of n and D_i ($i = 1$ to 24) are exactly the same as those of Equations (12) and (13).

Because the mass, damping and stiffness matrices, $[m^{(p)}]$, $[c^{(p)}]$ and $[k^{(p)}]$, of the moving mass element for plate are time-dependent, so are the overall mass, damping and stiffness matrices, $[\overline{M}]$, $[\overline{C}]$ and $[\overline{K}]$, of the entire structural system.

3.2 Overall external force vector

Figure 2 shows a concentrated mass m_c moves, with acceleration a, on a rectangular plate. If, at any instant of time *t*, the concentrated mass locates at the position $\overline{x}_m^{(p)}(t)$ and $\overline{y}_m^{(p)}(t)$ of the plate, then the external forces on all the nodes of the plate are equal to zero except the four nodes of the s_p^{th} plate element at which the concentrated mass is located at time *t*.

$$\{\overline{F}(t)\} = [0 \ \cdots \ f_{s_1}^{(p)}(t) \ \ f_{s_2}^{(p)}(t) \ \ f_{s_3}^{(p)}(t) \ \cdots \ f_{s_{24}}^{(p)}(t) \ \cdots \ 0]^T \tag{16}$$

where

$$f_{s_i}^{(p)}(t) = \Phi_i^{(p)} m_c a \cos\theta \quad (i = 1,7,13,19) \tag{17a}$$

$$f_{s_i}^{(p)}(t) = \Phi_i^{(p)} m_c a \sin\theta \quad (i = 2,8,14,20) \tag{17b}$$

$$f_{s_i}^{(p)}(t) = \Phi_i^{(p)} m_c g \qquad (i = 3, 4, 5, 9, 10, 11, 15, 16, 17, 21, 22, 23) \tag{17c}$$

$$f_{s_i}^{(p)}(t) = 0 \qquad (i = 6, 12, 18, 24) \tag{17d}$$

In Equation (16), $f_{s_i}^{(p)}(t)$ (i =1 to 24) are the equivalent nodal forces on the 24 degrees of freedom of the s_p^{th} plate element. In Equation (17), Equations (17a) and (17b) are those due to the horizontal ($\overline{x}$) and horizontal ($\overline{y}$) inertia forces of the moving load, respectively; while Equation (17c) are those due to the external load $m_c g$ induced by the concentrated mass m_c and located at the local coordinates [x , y]=[$x_m^{(p)}(t)$, $y_m^{(p)}(t)$] of the s_p^{th} plate element. It is noted that the symbols g

and θ appearing in the last expressions are respectively the gravity acceleration and the angle between the direction of moving concentrated mass m_c and the $\bar{x}$ axis (see Figure 2).

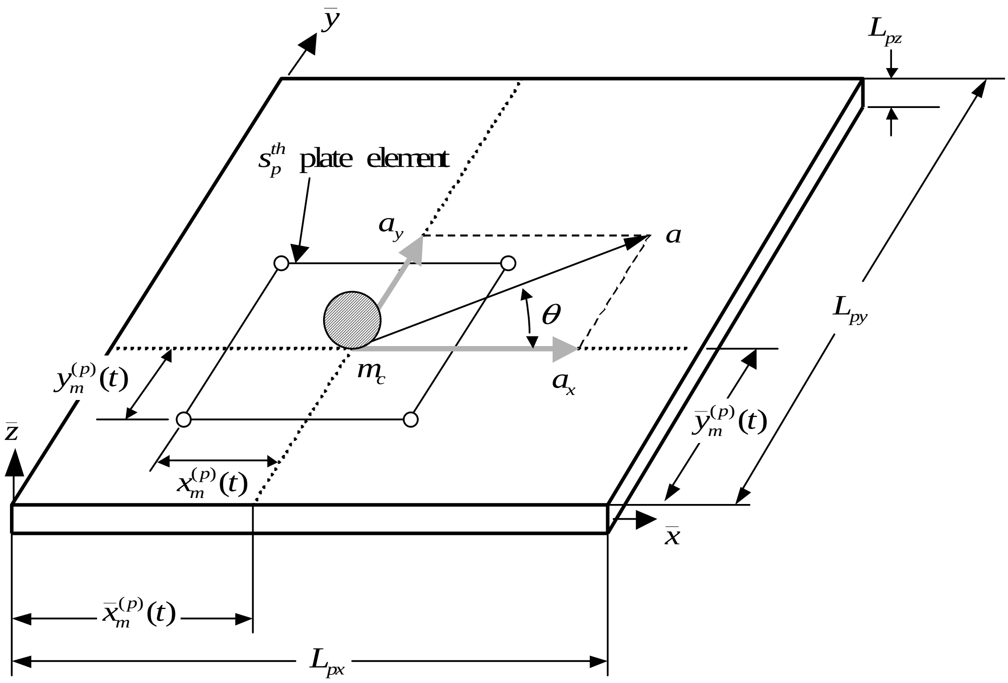


FIGURE 2 A CONCENTRATED MASS m_c MOVES, WITH ACCELERATION a, ON A FLAT PLATE.

3.3 Solution of equations of motion

In the present study, the dynamic responses of the flat plate due to a moving load, with the effects of inertial force, Coriolis force and centrifugal force considered, are determined by solving the equations of motion of the entire structural system given by Equation (10). The solution procedures are described in the following.

1. Since the initial conditions of the structural system are assumed to be "at rest" in this paper, $\{\bar{q}(0)\} = \{0\}$, $\{\dot{\bar{q}}(0)\} = \{0\}$ and $\{\ddot{\bar{q}}(0)\} = \{0\}$.
2. Using Equations (8a)-(8c) to calculate the property matrices of the moving mass element for plate, $[m^{(p)}]$, $[c^{(p)}]$ and $[k^{(p)}]$, at any time $t_{i+1} = t_i + \Delta t$ (i =0,1,2,…). Where Δt is time interval used for calculating the dynamic responses of the structure and taken to be 0.001s in this paper.
3. Using Equations (11)-(13), and imposing the prescribed boundary conditions, to determine the overall mass and stiffness matrices of the entire structural system, $[\bar{M}]$ and $[\bar{K}]$, at any time $t_{i+1} = t_i + \Delta t$ (i =0,1,2,…).
4. Determine the natural frequencies, ω_i (i =1,2,…), of the entire structural system by means of Jacobi algorithm [16].
5. Perform the Rayleigh damping theory (see Equations (14a)-(14c)) to determine the overall damping matrix, $[C_p]$, of the plate itself and calculate the overall damping matrix, $[\bar{C}]$, of the entire structural system (see Equation (15)).
6. Evaluate the overall equivalent nodal force vector (see Equations (16)-(17)) of the structure due to moving load at any time $t_{i+1} = t_i + \Delta t$ (i =0,1,2,…).
7. Solve the equations of motion of the entire structural system, Equation (10), with Newmark direct integration method [16] for its dynamic responses at any time $t_{i+1} = t_i + \Delta t$ (i =0,1,2,…).
8. Repeat steps 2-7 to obtain the dynamic responses of the structural system at any time $t_{i+1} = t_i + \Delta t$ (i =0,1,2,…).

Because the property matrices, $[m^{(p)}]$, $[c^{(p)}]$ and $[k^{(p)}]$, of the moving mass element for plate are time-variant, so are the overall mass, damping and stiffness matrices of the entire structural system. In such a case, the overall mass, damping and stiffness matrices must be calculated at each time step (see steps 3-5). For this reason, the time required for the computer calculation by the present technique is greater than that required by the existing literature without considering the inertial force, Coriolis force and centrifugal force of the moving load(s). However, it is believed that this should be the cost that one should pay if more satisfactory results are hoped to obtain.

IV. DYNAMIC RESPONSES OF THE BEAM DUE TO A MOVING LOAD

The dynamic responses of the flat plate due to a moving load can be determined by means of the formulations presented in sections 2-3 of this paper. Because the vibration responses of the flat plate subjected to a moving load should be close to those of the beam, with its sizes being the same as those of the flat plate, subjected to the same loading conditions [2], the formulations for calculating the dynamic responses of the beam due to a moving load are also presented in this section. Thus, the vibration characteristics of the rectangular plate due to a moving load obtained from the formulations of sections 2-3 can be validated by using those of the beam obtained from the formulations of this section.

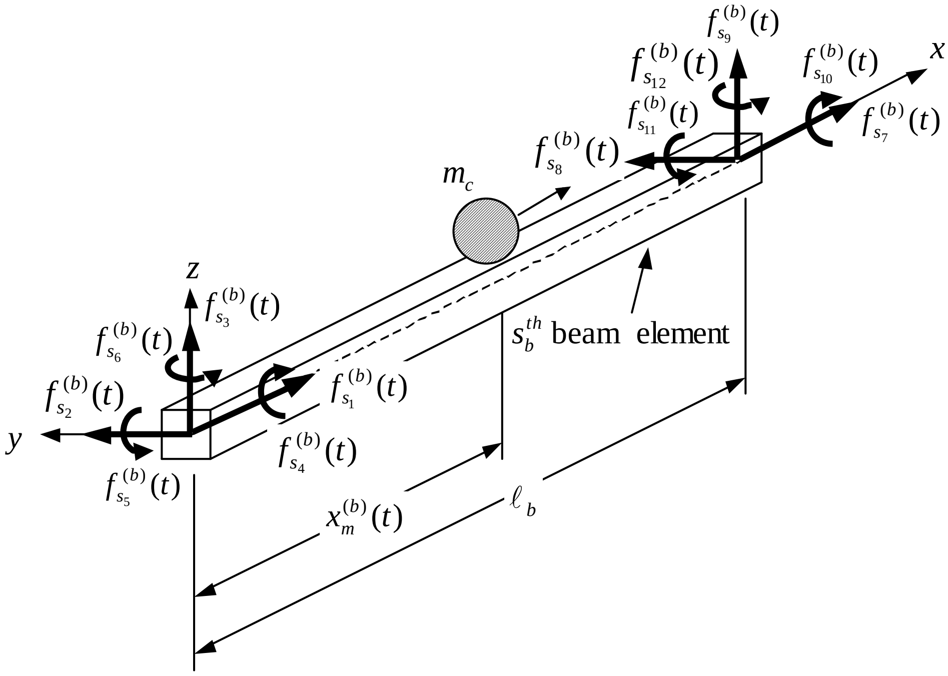


FIGURE 3 THE s_b^{th} BEAM ELEMENT SUBJECTED TO A MOVING LOAD AT THE INSTANT OF TIME t.

Since each beam element consists of two nodes and 12 degrees of freedom (see Figure 3), the mass, damping and stiffness matrices, $[m^{(b)}]$, $[c^{(b)}]$ and $[k^{(b)}]$, of the moving mass element *for beam* are 12×12 matrices and can be derived by using the similar procedures of section 2, where all the coefficients of the above-mentioned matrices are equal to zero except that

$$m_{ij}^{(b)} = m_c \Phi_i^{(b)} \Phi_j^{(b)} \quad (i, j = 1,7) \tag{18a}$$

$$m_{ij}^{(b)} = m_c \Phi_i^{(b)} \Phi_j^{(b)} \quad (i, j = 2,6,8,12) \tag{18b}$$

$$m_{ij}^{(b)} = m_c \Phi_i^{(b)} \Phi_j^{(b)} \quad (i, j = 3,5,9,11) \tag{18c}$$

$$c_{ij}^{(b)} = 2 m_c V_{mx} \Phi_i^{(b)} \Phi_j^{(b)\prime} \quad (i, j = 2,6,8,12) \tag{18d}$$

$$c_{ij}^{(b)} = 2 m_c V_{mx} \Phi_i^{(b)} \Phi_j^{(b)\prime} \quad (i, j = 3,5,9,11) \tag{18e}$$

$$k_{ij}^{(b)} = m_c V_{mx}^2 \Phi_i^{(b)} \Phi_j^{(b)\prime\prime} \quad (i, j = 2,6,8,12) \tag{18f}$$

$$k_{ij}^{(b)} = m_c V_{mx}^2 \Phi_i^{(b)} \Phi_j^{(b)''} \quad (i, j = 3,5,9,11) \tag{18g}$$

where V_{mx} is the velocity of the moving concentrated mass m_c in the $\bar{x}$ direction; while $\Phi_i^{(b)}(t)$ (i =1 to 4) are the shape functions with non-zero ones given by [15]

$\Phi_1^{(b)} = 1 - \varsigma_b$, $\Phi_2^{(b)} = \Phi_3^{(b)} = 1 - 3\varsigma_b^{\ 2} + 2\varsigma_b^{\ 3}$, $\Phi_4^{(b)} = \Phi_{10}^{(b)} = 0$,

$\Phi_5^{(b)}(t) = \Phi_6^{(b)}(t) = [\varsigma_b - 2\varsigma_b^{\ 2} + \varsigma_b^{\ 3}]\ell_b$, $\Phi_7^{(b)} = \varsigma_b$,

$$\Phi_8^{(b)} = \Phi_9^{(b)} = 3\varsigma_b^{\ 2} - 2\varsigma_b^{\ 3}, \ \Phi_{11}^{(b)} = \Phi_{12}^{(b)} = [-\varsigma_b^{\ 2} + \varsigma_b^{\ 3}]\ell_b \tag{19a}$$

$$\varsigma_b = x_m^{(b)}(t)/\ell_b \tag{19b}$$

where $x_m^{(b)}(t)$ is the distance between the position of the moving mass m_c and the left-end of the beam element on which it applies; while ℓ_b is the length of the beam element (see Figures 3 and 4).

To determine the overall property matrices of the entire structural system, one requires to add the property matrices of the moving mass element for beam and those of the entire beam structure together, i.e.,

$$[\overline{M}] = [M_b] + [m^{(b)}]_{12\times12} \tag{20a}$$

$$[\overline{C}] = [C_b] + [c^{(b)}]_{12\times12} \tag{20b}$$

$$[\overline{K}] = [K_b] + [k^{(b)}]_{12\times12} \tag{20c}$$

where

$$\overline{M}_{ij} = M_{b,ij} \quad (i, j = 1 \text{ to } n) \tag{21a}$$

$$\overline{C}_{ij} = C_{b,ij} \quad (i, j = 1 \text{ to } n) \tag{21b}$$

$$\overline{K}_{ij} = K_{b,ij} \quad (i, j = 1 \text{ to } n) \tag{21c}$$

except that

$$\overline{M}_{D_iD_j} = M_{b,D_iD_j} + m_{ij}^{(b)} \quad (i, j = 1 \text{ to } 12) \tag{22a}$$

$$\overline{C}_{D_iD_j} = C_{b,D_iD_j} + c_{ij}^{(b)} \quad (i, j = 1 \text{ to } 12) \tag{22b}$$

$$\overline{K}_{D_iD_j} = K_{b,D_iD_j} + m_{ij}^{(b)} \quad (i, j = 1 \text{ to } 12) \tag{22c}$$

In the last expressions, $[M_b]$, $[C_b]$ and $[K_b]$ are the overall mass, damping and stiffness matrices of the beam itself, respectively. In which, the overall damping matrix, $[C_b]$, is obtained by means of the Rayleigh damping theory [15] (cf. Equation (14)). Moreover, D_i ($i = 1$ to 12) respectively represent the numberings for the 12 degrees of freedom of the two nodes of the D^{th} beam element on which the moving load applies at time *t*.

If, at any instant of time *t*, the concentrated mass m_c is located at the position $\bar{x}_m^{(b)}(t)$ of the beam (see Figure 4), then the overall external force vector of the entire beam induced by the concentrated mass m_c takes the form

$$\{\overline{F}(t)\} = [0 \quad 0 \quad \cdots \quad f_{s_1}^{(b)}(t) \quad f_{s_2}^{(b)}(t) \quad \cdots \quad f_{s_{11}}^{(b)}(t) \quad f_{s_{12}}^{(b)}(\mathrm{t}) \quad \cdots \quad 0 \quad 0]^T \tag{23}$$

where

$$f_{s_i}^{(b)}(t) = m_c g \Phi_i^{(b)} \quad (i = 3, 5, 9, 11) \tag{24a}$$

$$f_{s_i}^{(b)}(t) = 0 \ \ (i = 1, 2, 4, 6, 7, 8, 10, 12) \tag{24b}$$

It is worthy of mentioned that the symbol g refers to the acceleration of gravity and the subscripts, s_i ($i = 1$ to 12) in Equations (23)-(24) represent the 12 degrees of freedom of the s_b^{th} beam element at which the concentrated mass m_c is located.

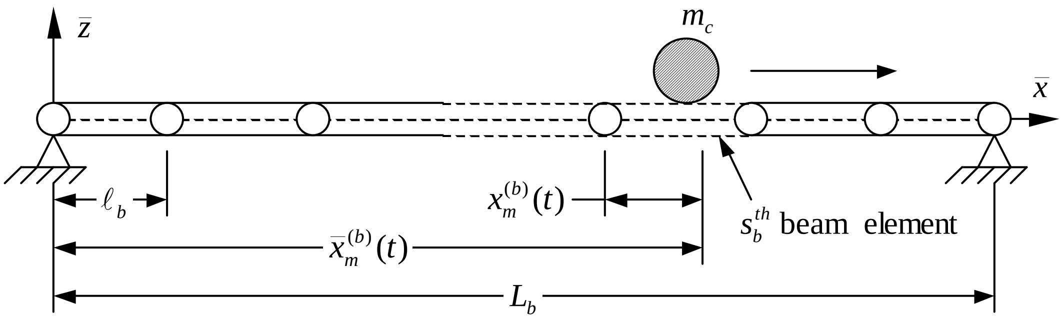


FIGURE 4 A BEAM SUBJECTED TO A MOVING LOAD.

Finally, one may use the similar procedures present in section 3.3 to solve the equations of motion, Equation (10), of the structural system for the vibration responses of the beam due to a moving load.

V. NUMERICAL RESULTS AND DISCUSSIONS

In this section, one plate model and one beam model, respectively, subjected to a moving load are studied. In which, the plate is made of steel with mass density ρ_p = 7820 kg/m^3, modulus of elasticity E_p = 206.8 GN/m^2 and Poisson's ratio ν = 0.29, while its dimensions are: length L_{px} = 1.0 m, width L_{py} = 0.5 m and thickness L_{pz} = 0.01 m (see Figure 5). The entire plate is modelled with 32 identical 0.125 $m \times$ 0.125 m rectangular plate elements and 45 nodes. For conveniences, the last rectangular plate with side AB and side CD being constrained as pin joints is called Pin-plate.

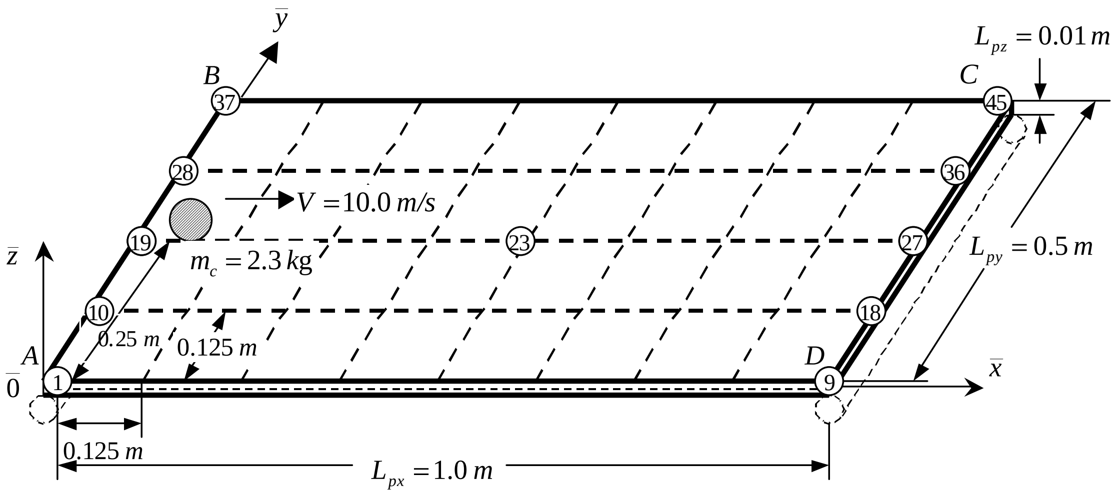


FIGURE 5 A PINNED-PINNED RECTANGULAR PLATE (PIN-PLATE) SUBJECTED TO A MOVING LOAD WITH MASS $m_c =$ 2.3 KG AND A CONSTANT SPEED V =10.0 M/S.

Since the vibration characteristics of the rectangular plate, with its bending effects neglected (i.e., Poisson's ratio $\nu \approx 0$), will be close to those of the beam [2], a beam model is also studied in order to validate the availability of the presented theory. The beam model corresponding to Pin-plate is called Pin-beam, where the dimensions and material properties of Pin-beam are exactly the same as those of Pin-plate (i.e., length L_b = 1.0 m and cross-sectional area 0.5 $m \times$ 0.01 m, mass density ρ_b = 7820 kg/m^3, modulus of elasticity E_b = 206.8 GN/m^2). In addition, the finite element model for Pin-beam is composed of 9 nodes and 8 beam elements (see Figure 6). Three values of Poisson's ratio ($\nu = 0.0$, 0.15, 0.29) for Pin-plate were studied, however, according to the beam theory, the Poisson's ratio for Pin-beam is equal to zero. Unless specially stated, the damping ratios (ξ_i) used for the plate model and the beam model of this paper are taken to be 0.005 and the time interval used for calculating the dynamic responses of the entire structural system is 0.001s, i.e., $\Delta t = 0.001$ s.

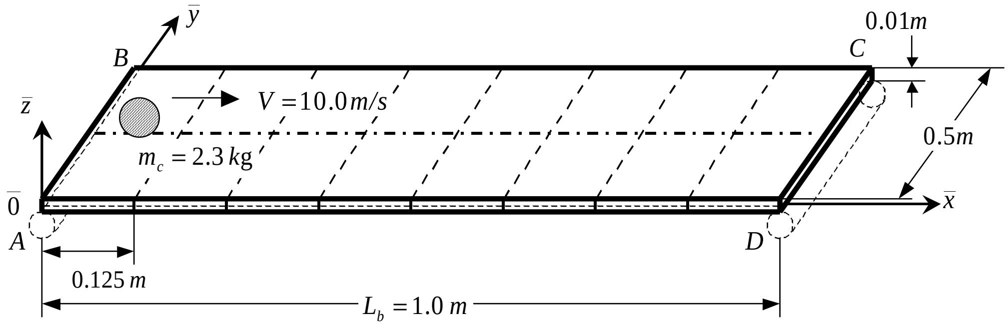


FIGURE 6 A PINNED-PINNED BEAM SUBJECTED TO A MOVING LOAD WITH MASS $m_c =$ 2.3 KG AND A CONSTANT SPEED V =10.0 M/S.

5.1 Validation

The lowest few natural frequencies of Pin-plate (with Poisson's ratio $\nu = 0.0$, 0.15 and 0.29, respectively) and Pin-beam are listed in columns 2, 3, 4 and 5 of Table 1. Because the total degree of freedom of Pin-plate is much more than that of Pin-beam, one can find only four of the lowest ten natural frequencies of Pin-plate to be corresponding to the lowest four natural frequencies of the Pin-beam, as shown in the fifth column of Table 1. It is noted that the correspondence between the natural frequencies of Pin-plate and those of Pin-beam must be made based on their corresponding mode shapes. From the table, it can be found that the natural frequencies of Pin-plate will be close to the corresponding ones of Pin-beam if the Poisson's ratio (ν) approaches zero. Based on this result, one may infer that the central displacements of Pin-plate subjected to a moving load will be close to those of Pin-beam subjected to the same loading conditions if the Poisson's ratio (ν) of the plate approaches zero.

TABLE 1
THE LOWEST FEW NATURAL FREQUENCIES ω_i (Hz) OF PIN-PLATE AND PIN-BEAM.

Mode No. for Pin-Plate	Pin-plate			Pin-Beam
	ν = 0.29	**ν = 0.15**	**ν = 0.0**	
1st	23.6032	23.4487	23.3869	23.3150
2nd	63.1528	66.1188	70.0181	-----
3rd	96.3655	94.9433	94.4052	93.2379
4th	146.4804	150.3048	155.8149	-----
5th	215.1707	212.2335	213.6403	-----
6th	221.5174	217.402	215.8878	209.8451
7th	264.8742	267.1878	272.0683	-----
8th	283.4873	285.2118	292.2830	-----
9th	380.7570	385.1473	-----	-----
10th	403.5357	395.9960	393.2355	373.7035

If a load with mass m_c = 2.3 *kg* moves with a constant speed V_c = 10.0 *m/s* from side AB to side CD, along the centrelines, of Pin-plate (see Figure 5) and Pin-beam (see Figure 6), then the vertical ($\bar{z}$) central displacements of Pin-plate and Pin-beam are shown in Figure 7, where the solid line (—), the solid line with triangles (—Δ—) and the solid line with circles (—○—) represent the vertical ($\bar{z}$) central displacements of Pin-plate with Poisson's ratio ν = 0.0, 0.15 and 0.29, respectively, while the dashed line (----) represents those of Pin-beam. From the figure, it is seen that the curves for Pin-plate are close to the curve for Pin-beam when the value of Poisson's ratio (ν) approaches zero. This agrees with the numerical results presented in reference [2].

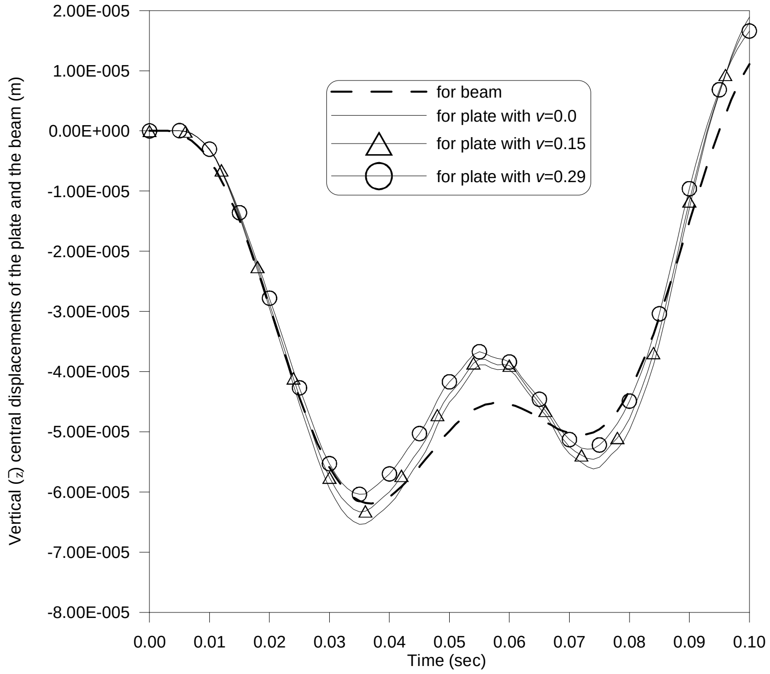


FIGURE 7 VERTICAL ($\bar{z}$) CENTRAL DISPLACEMENTS OF PIN-PLATE AND PIN-BEAM.

From all the numerical results presented in this subsection, it is believed that the presented theory is available for the title problem and will be used for further studies in this research.

5.2 Influence of moving-load speed

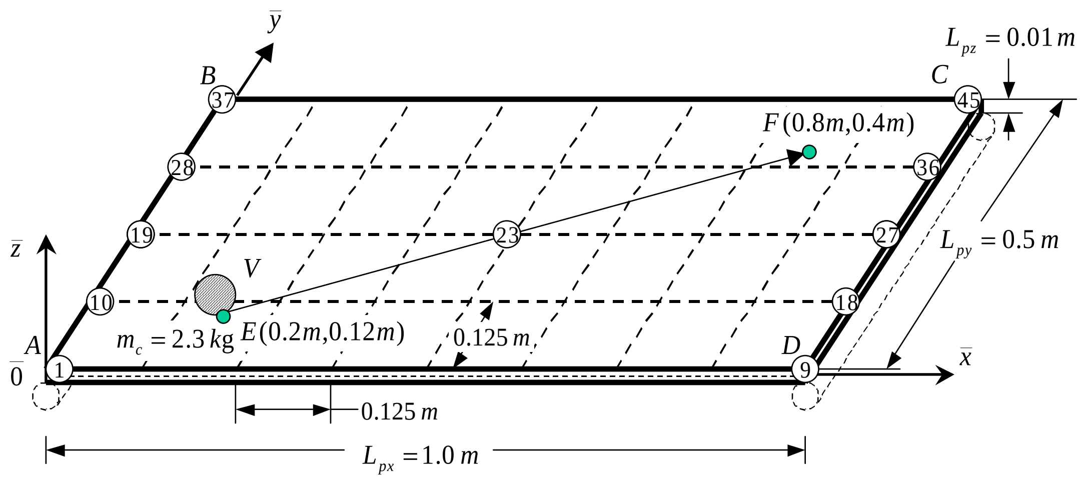


FIGURE 8 A PINNED-PINNED PLATE (PIN-PLATE) SUBJECTED TO A MOVING LOAD WITH MASS m_c = 2.3 KG AND MOVING SPEED V.

In this subsection, a load of mass m_c =2.3 *kg* moves with a constant speed V from point E ($\bar{x}_1$ =0.2*m*, $\bar{y}_1$ =0.12*m*) to point F ($\bar{x}_2$ =0.8*m*, $\bar{y}_2$ =0.4*m*) of the Pin-plate (see Figure 8) is investigated. Figure 9 shows the time histories for the vertical ($\bar{z}$) displacements of the centre point of Pin-plate, where the solid curves with circles (—○—) represent the time histories with moving-load speed $V = 5.0m/s$, those with crosses (—+—) represent the ones with $V = 10.0m/s$ and those with triangles (—Δ—) represent the ones with $V = 20.0m/s$. From the figure, one sees that the larger the moving-load speed, the larger the maximum vertical ($\bar{z}$) central displacements of the flat plate.

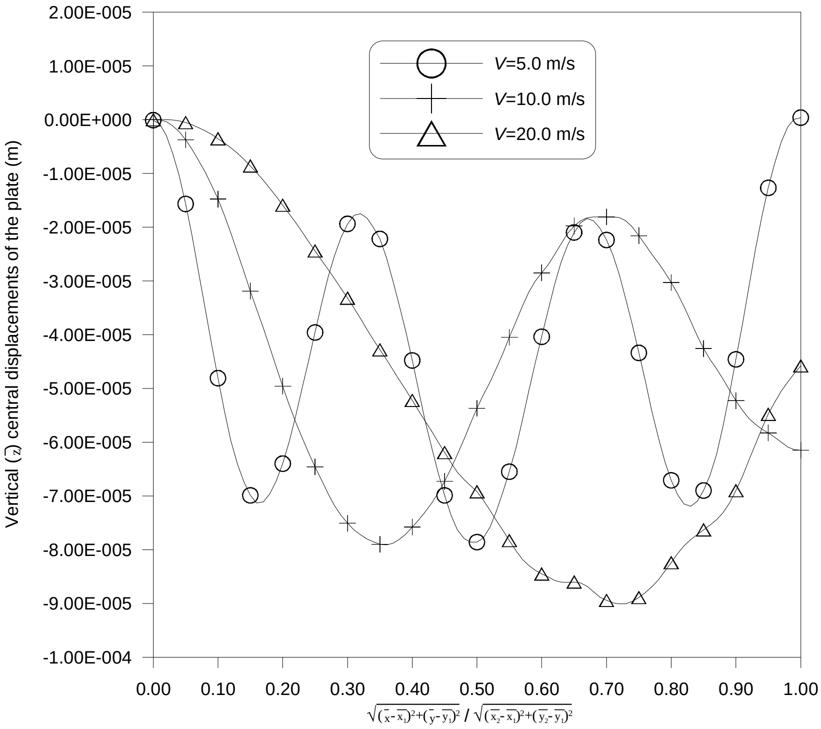


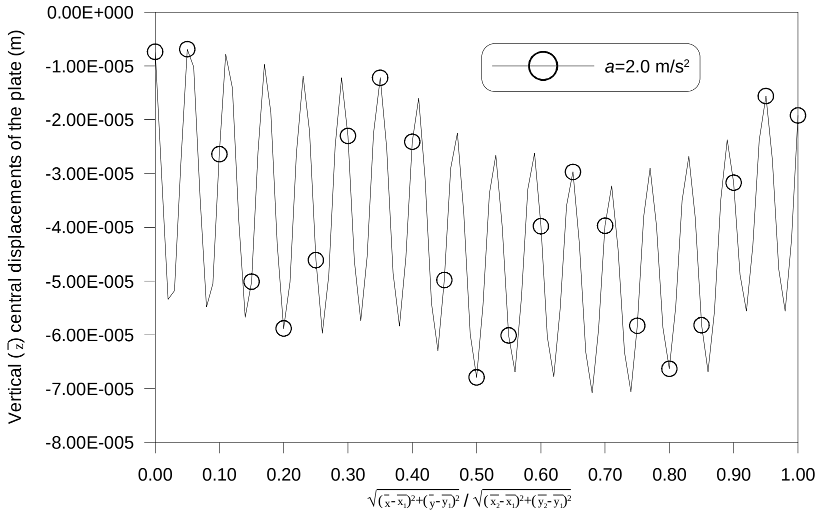
FIGURE 9 VERTICAL ($\bar{z}$) CENTRAL DISPLACEMENTS OF PIN-PLATE DUE TO A MOVING LOAD WITH A CONSTANT SPEED: (A) V = 5.0 M/S, (B) V =10.0 M/S AND (C) V =20.0 M/S.

5.3 Influence of acceleration

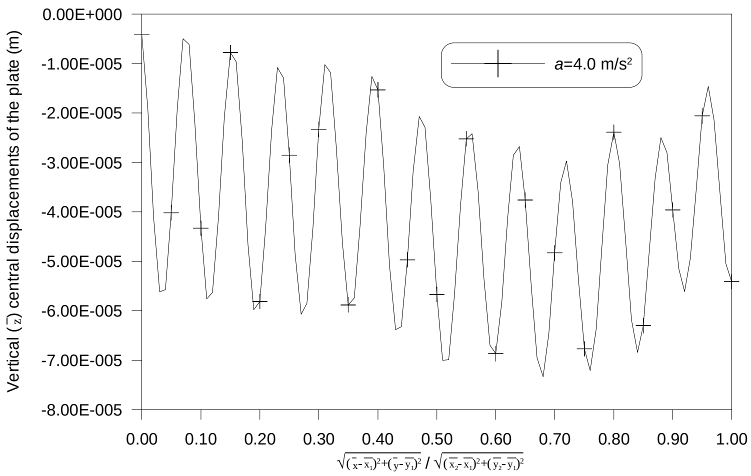
If the mass m_c moves, with initial velocity V =0.0 *m/s* and constant accelerations a =1.0, 4.0 and 8.0 m/s^2, from point E ($\bar{x}_1$ =0.2*m*, $\bar{y}_1$ =0.12*m*) to point F ($\bar{x}_2$ =0.8*m*, $\bar{y}_2$ =0.4*m*) of the Pin-plate, then the time histories for the vertical ($\bar{z}$) displacements of the centre point of Pin-plate are shown in Figures 10(a), 10(b) and 10(c), respectively. In which, the solid curves with circles (—O—), the solid curve with crosses (—+—) and the solid curves with triangles (—△—) represent the responses corresponding to the accelerations a =1.0, 4.0 and 8.0 m/s^2, respectively. From the figures, one finds that the larger the acceleration of the moving mass m_c, the larger the maximum vertical ($\bar{z}$) displacement of the centre point of Pin-plate. Since the acceleration of the moving mass has a close relation with its velocity and the latter significantly affects the vertical ($\bar{z}$) responses of the Pin-plate, one must also consider the effect of velocity in addition to the acceleration.

5.4 Influence of inertia force

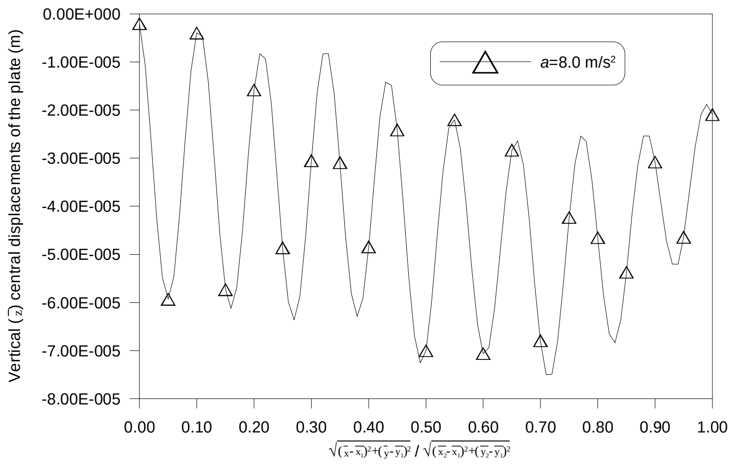
From the formulation of this paper, one can see that the effect of inertia force induced by the moving mass is to appear in the mass matrix $[m^{(p)}]$ of the moving mass element for plate as one may see from Equation (8a). Hence, if the mass matrix of the moving mass element is taken to be zero, i.e., $[m^{(p)}]$ = [0], then the effect of inertia force due to the moving mass will disappear.



(A)

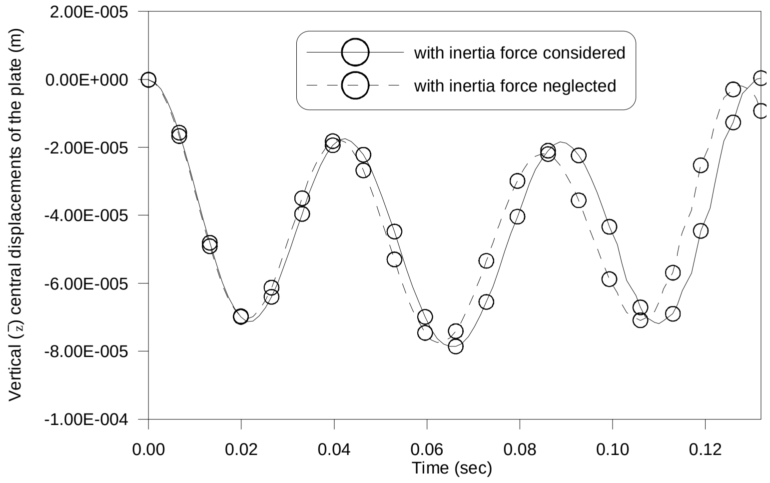


(B)

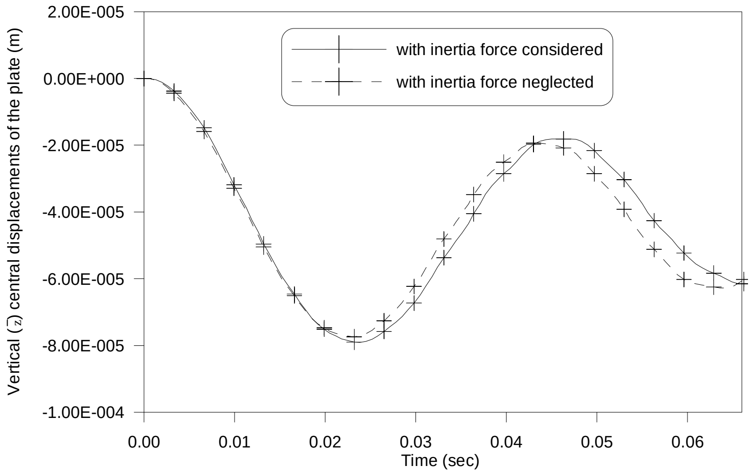


(C)

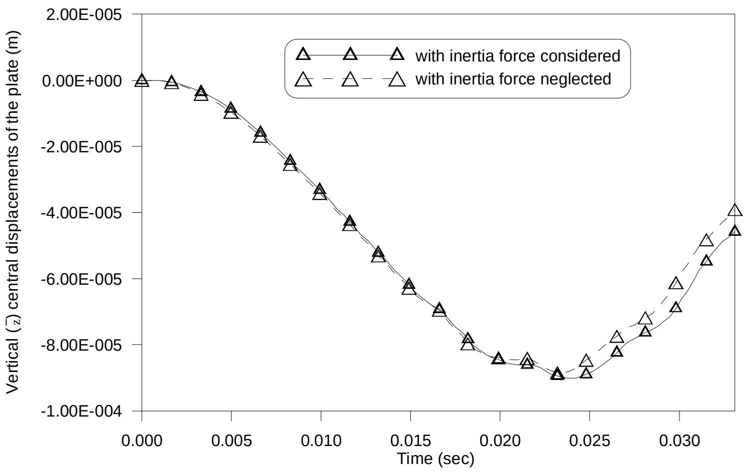
FIGURE 10 INFLUENCE OF ACCELERATION ON THE VERTICAL ($\bar{z}$) CENTRAL DISPLACEMENTS OF PIN-PLATE DUE TO A MOVING LOAD WITH A CONSTANT ACCELERATION: (A) a = 2.0 m/s^2, (B) a =4.0 m/s^2 AND (C) a =8.0 m/s^2.



(A)

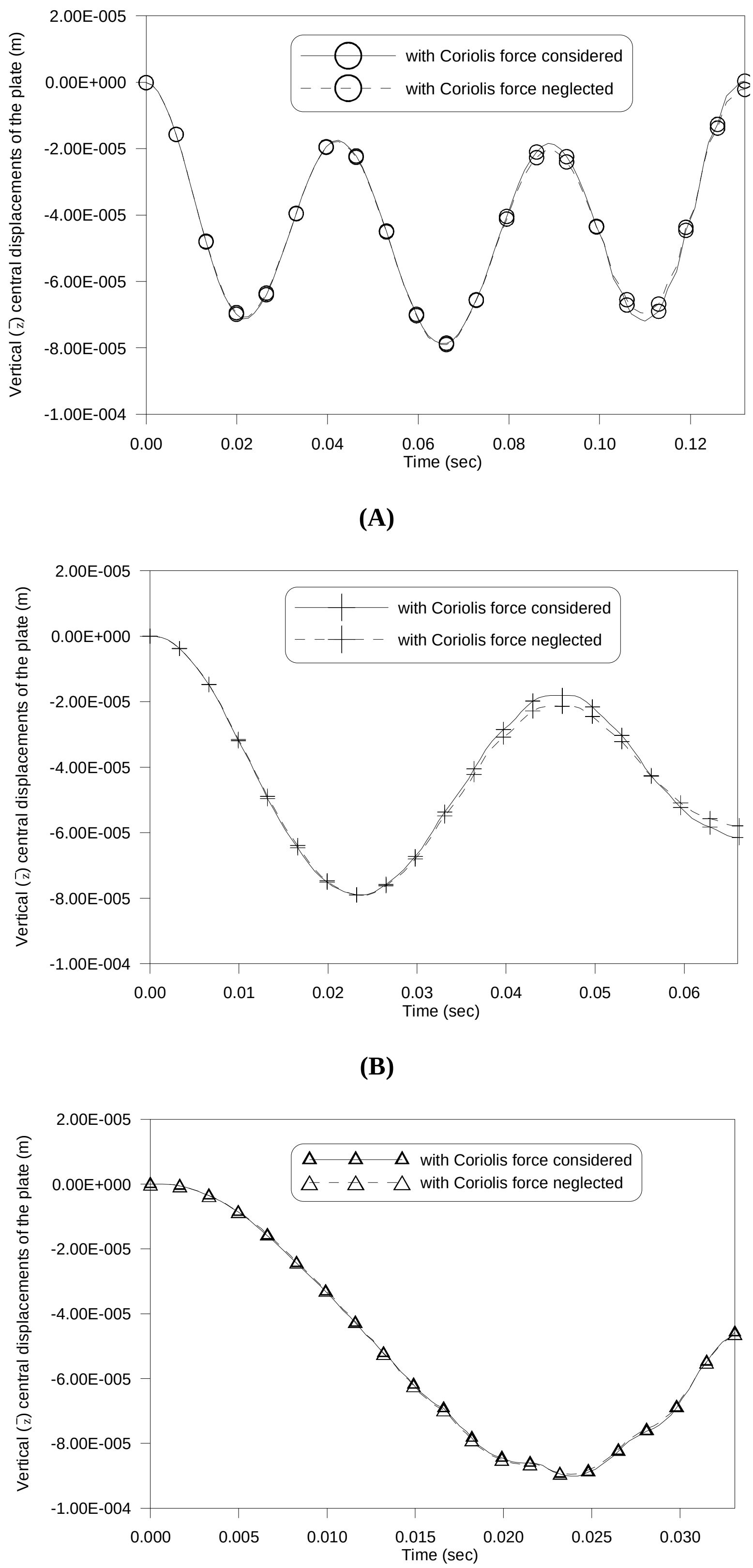


(B)



(C)

FIGURE 11 INFLUENCE OF INERTIA FORCE ON THE VERTICAL ($\bar{z}$) CENTRAL DISPLACEMENTS OF PIN-PLATE DUE TO A MOVING LOAD WITH A CONSTANT SPEED: (A) V = 5.0 M/S, (B) V =10.0 M/S AND (C) V =20.0 M/S.

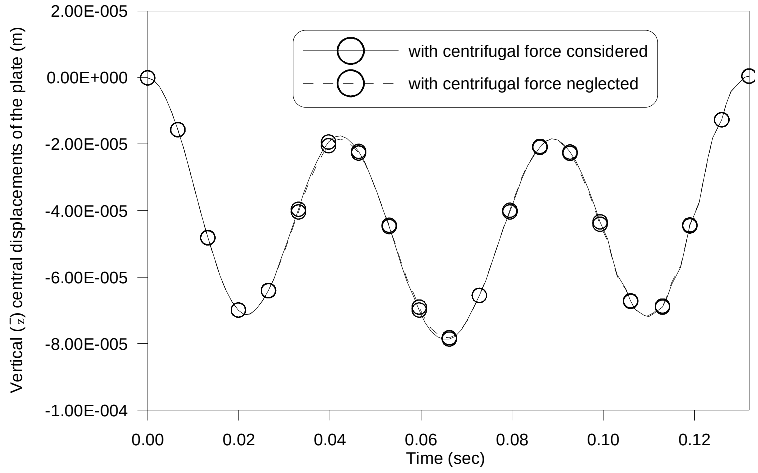


(A)

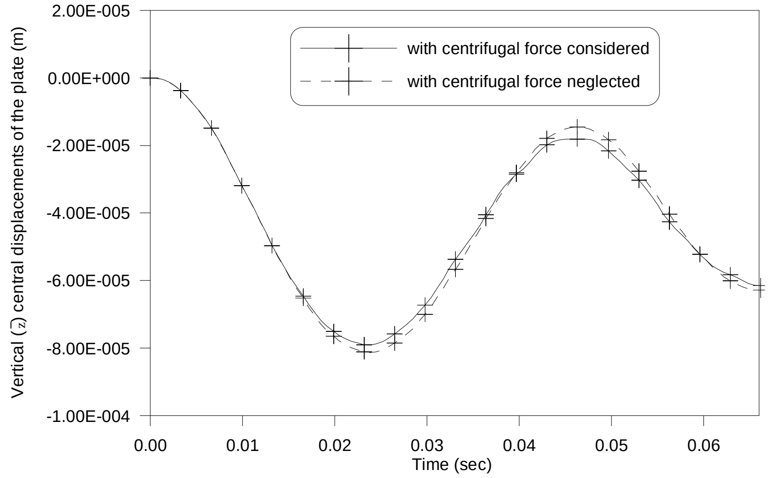
(B)

(C)

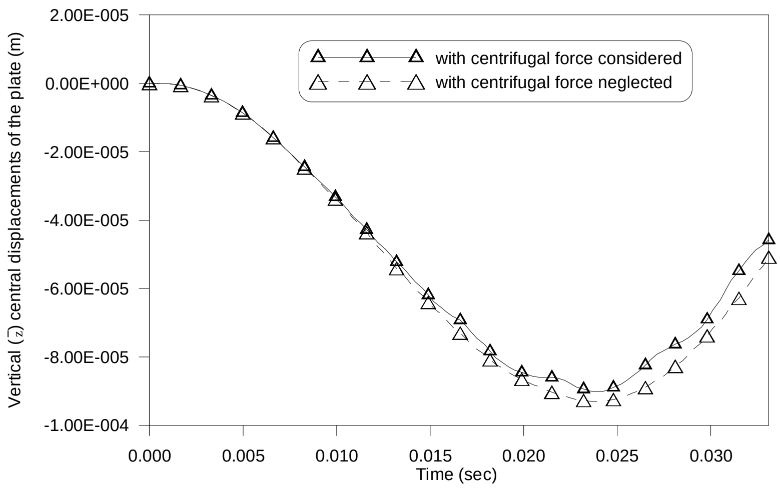
FIGURE 12 INFLUENCE OF CORIOLIS FORCE ON THE VERTICAL ($\bar{z}$) CENTRAL DISPLACEMENTS OF PIN-PLATE DUE TO A MOVING LOAD WITH A CONSTANT SPEED: (A) V = 5.0 M/S, (B) V =10.0 M/S AND (C) V =20.0 M/S.



(A)



(B)



(C)

FIGURE 13 INFLUENCE OF CENTRIFUGAL FORCE ON THE VERTICAL ($\bar{z}$) CENTRAL DISPLACEMENTS OF PIN-PLATE DUE TO A MOVING LOAD WITH A CONSTANT SPEED: (A) V = 5.0 M/S, (B) V =10.0 M/S AND (C) V =20.0 M/S.

Figure 11 shows the time histories for the vertical ($\bar{z}$) displacements of the centre point of Pin-plate when the mass m_c =2.3 *kg* moves with a constant speed V from point E ($\bar{x}_1$ =0.2*m*, $\bar{y}_1$ =0.12*m*) to point F ($\bar{x}_2$ =0.8*m*, $\bar{y}_2$ =0.4*m*) of the flat plate. In which, the curves with circles (—○— and --○--), crosses (—+— and --+--) and triangles (—Δ— and --Δ--) represent the time histories with moving-load speed $V = 5.0m/s$, $V = 10.0m/s$ and $V = 20.0m/s$, respectively. Besides, the solid and dashed curves represent those with the effect of inertia force of moving load considered and neglected. From the figure, one sees that the maximum vertical ($\bar{z}$) central displacement with the effect of inertia force considered is larger than that with the effect of inertia force neglected. Thus, the effect of inertia force is important and should be considered in the formulations.

5.5 Influence of Coriolis force

Similarly, one can ignore the effect of the Coriolis force due to moving load by taking the damping matrix of the moving mass element for plate to be zero, i.e., $[c^{(p)}]$ = [0]. The same example as that of the last subsection is studied and the vertical ($\bar{z}$) displacements of the centre point of the flat plate are shown in Figure 12. The legends for the curves in the figure are exactly the same as those in Figure 11 except that the inertia force is replaced by the Coriolis force. From the figure, one sees that the Coriolis force affects the vertical ($\bar{z}$) central displacements of the plate to some degree.

5.6 Influence of centrifugal force

In this subsection, the effect of centrifugal force due to moving load is ignored by taking the stiffness matrix of the moving mass element for plate to be zero, i.e., $[k^{(p)}]$ = [0]. The same plate as that of the last subsection is investigated and the vertical ($\bar{z}$) and central displacements of the Pin-plate are shown in Figure 13. The legends for the curves are exactly the same as those in Figure 12 except that the Coriolis force is replaced by the centrifugal force.

From Figure 13, one sees that the influence of the centrifugal force on the vertical ($\bar{z}$) central displacements of the plate increases with increasing the moving-load speed. This is because the magnitude of the centrifugal force appearing in the stiffness matrix $[k^{(p)}]$ of the moving mass element for plate is proportional to the square of the moving-load speed (see Equation (8c)).

VI. CONCLUSION

1. To take account of the effects of inertia force, Coriolis force and centrifugal force of the moving load, the theory of the moving mass element *for plate* and that *for beam* are presented. In which, the property matrices of the moving mass elements are derived based on the superposition principle and the definition of shape functions. It is found that the order of the property matrices of the moving mass element *for plate* is 24×24 , while that *for beam* is 12×12 .
2. Combination of the property matrices of the moving mass element and the overall property matrices of the plate (or beam) itself gives the overall property matrices of the entire structural system. Because the property matrices of the moving mass element for plate and that for beam have something to do with the instantaneous position of the moving load, both the property matrices of the moving mass element and the entire structural system are time-variant.
3. The moving speed, acceleration, inertia force, Coriolis force and centrifugal force of the moving load have significant influences on the vertical ($\bar{z}$) dynamic responses of the flat plate. Thus, all the above-mentioned parameters should be considered in the formulations.

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