Reliability Evaluation of Multicommodity Limited-Flow Networks with Budget Constraints

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***Abstract*—*Many real-world systems such as manufacturing systems, transportation systems and logistics/distribution systems that play important roles in our modern society can be regarded as multicommodity flow networks whose arcs have independent, finite and multi-valued random capacities. Such a flow network is a multistate system with multistate components and its reliability for level (**d**;c), i.e., the probability that k different types of commodity can be transmitted from the source node to the sink node such that the demand level $\mathbf{d}=(d_1,d_2,...,d_k)$ is satisfied and the total transmission cost is less than or equal to c, can be evaluated in terms of minimal path vectors to level (**d**;c) (named (**d**;c)-MPs here). The main objective of this paper is to present an intuitive algorithm to generate all (**d**;c)-MPs of such a flow network for each level (**d**;c) in terms of minimal pathsets. Two examples are given to illustrate how all (**d**;c)-MPs are generated by our algorithm and then the reliability of one example is computed.*

***Keywords*— Reliability, limited-flow network, multicommodity, multistate system, (d;c)-MP.**

I. INTRODUCTION

Reliability is an important performance indicator in the planning, designing, and operation of a real-world system. Traditionally, it is assumed that the system under study is represented by a probabilistic graph in a binary-state model, and the system operates successfully if there exists one or more paths from the source node s to the sink node t. In such a case, reliability is considered as a matter of connectivity only and so it does not seem to be reasonable as a model for some real-world systems. Many physical systems such as manufacturing systems, transportation systems, and logistics/distribution systems can be regarded as flow networks in which arcs have independent, finite, and integer-valued random capacities. To evaluate the system reliability of such a flow network, different approaches have been presented [7, 9, 14-23, 26-28].. However, these models have assumed that the flow along any arc consisted of a single commodity only. For such a flow network with multicommodity, it is very practical and desirable to compute its reliability for level (**d**;c), i.e., the probability that k different types of commodity can be transmitted from the source node to the sink node in the way that the demand level $\mathbf{d}=(d_1,d_2,...,d_k)$ is satisfied and the total transmission cost is less than or equal to c.

In general, reliability evaluation can be carried out in terms of minimal pathsets (MPs) in the binary state model case and (**d**;c)-MPs (i.e., minimal path vectors to level (**d**;c) [3], lower boundary points of level (**d**;c) [12], or upper critical connection vector to level (**d**;c) [7]) for each level (**d**;c) in the multistate model case. The multicommodity limited-flow network with budget constraints here can be treated as a multistate system of multistate components and so the need of an efficient algorithm to search for all of its (**d**;c)-MPs arises. The main purpose of this article is to present a simple algorithm to generate all (**d**;c)-MPs of such a network in terms of minimal pathsets. Two examples are given to illustrate how all (**d**;c)-MPs are generated and the reliability of one example is calculated by further applying the state-space decomposition method [4].

II. BASIC ASSUMPTIONS

Let $G=(N,A,U)$ be a directed limited-flow network with the unique source s and the unique sink t, where N is the set of nodes, $A=\{a_i \mid 1\le i\le n\}$ is the set of arcs, and $U=(u_1,u_2,...,u_n)$, where u_i denotes the maximum capacity of each arc a_i for $i=1,2,...,n.$ Such a flow network is assumed to further satisfy the following assumptions:

1. Each node is perfectly reliable. Otherwise, the network will be enlarged by treating each of such nodes as an arc [1].
2. The capacity of each arc a_i is an integer-valued random variable that takes integer values from 0 to u_i according to a given distribution.

3. Every unit flow of commodity ℓ consumes a given amount ρ^{ℓ} of the capacity associated with each arc.
4. The capacities of different arcs are statistically independent.
5. Flow in the network must be integer-valued and satisfy the so-called flow-conservation law [10]. This means that no flow will disappear or be created during the transmission.

Assumption 4 is made just for convenience. If it fails in practice, the proposed algorithm to search for all (**d**;c)-MPs is still valid except that the reliability computation in terms of such (**d**;c)-MPs should take the joint probability distributions of all arc capacities into account.

Since there are k different types of commodity within the network, the system demand level can be represented as a k-tuple vector $\mathbf{d}=(d_1,d_2,...,d_k)$ where d_j is the demand level of commodity j for $j=1,2,...,k$. Let $X=(x_1,x_2,...,x_n)$ be a system-state vector (i.e., the current capacity of each arc a_i under X is x_i, where x_i takes integer values $0,1,2,...,u_i$), and $V(X)=(V(X)_1,V(X)_2,...,V(X)_k)$, the system maximal flow vector under X where $V(X)_j$ denotes the maximal flow of commodity j under X. (Whenever $k\geq 2$, there may be more than one maximal flow vector for each X. See the Appendix for more details.) Under the system-state vector $X=(x_1,x_2,...,x_n)$, the arc set A has the following three important subsets: $N_X=\{a_i\in A\,|\,x_i>0\}$, $Z_X=\{a_i\in A\,|\,x_i=0\}$, and $S_X=\{a_i\in N_X\,|\,V(X-e_i)<V(X)\}$, where $e_i=(\delta_{i1},\delta_{i2},...,\delta_{in})$, with $\delta_{ij}=1$ if $j=i$ and 0 if $j\neq i$. In fact, $A=S_X\cup(N_X\setminus S_X)\cup Z_X$ is a disjoint union of A under X.

A system-state vector X is said to be a (**d**;c)-MP if and only if: (1) its system capacity level is **d** (i.e., V(X)=**d**), (2) each nonzero-capacity arc under X is sensitive (*i.e.*, $N_X = S_X$), and (3) the total transmission cost is less than or equal to c. If level (**d**;c) is given, then the probability that *k* different types of commodity can be transmitted from the source node to the sink node in the way that the demand level $\mathbf{d}=(d_1,d_2,...,d_k)$ is satisfied and the total transmission cost is less than or equal to c, is taken as the system reliability.

III. MODEL BUILDING

Suppose that $P^1,P^2,...,P^m$ are the collection of all MPs of the system, and let $C=(c_1^1,c_1^2,...,c_1^k,c_2^1,c_2^2,...,c_2^k,...,c_n^1,c_n^2,...,c_n^k)$ denote the transmission cost vector where c_i^{ℓ} is the unit transmission cost of commodity ℓ through arc a_i. For each P^j, $W_j^{\ell}=\sum_i\{c_i^{\ell}\,|\,a_i\in P^j\}$ and $L_j=\min\{u_i\,|\,a_i\in P^j\}$ are taken as the unit transmission cost of commodity ℓ and maximum capacity through it, respectively. Under the flow-conservation law, any feasible flow pattern from s to t should satisfy that (1) the total flow-in and the total flow-out of each commodity for any given node (except for s and t) are equal, and (2) every unit flow of each commodity from s to t should travel through one of the MPs. Hence, under the system-state vector $X=(x_1,x_2,...,x_n)$ with $V(X)=(d_1,d_2,...,d_k)$, any feasible flow pattern that the total transmission cost is less than or equal to c can be represented as a flow vector $(f_1^1,f_1^2,...,f_1^k,f_2^1,f_2^2,...,f_2^k,...,f_m^1,f_m^2,...,f_m^k)$ where f_j^{ℓ} is the flow of commodity ℓ transmitted through P^j such that the following four conditions are satisfied:

$$\sum_{j=1}^{m} f_j^{\ell}=d_{\ell} \text{ for each } \ell=1,2,...,k \tag{1}$$

$$\sum_{\ell=1}^{k} f_j^{\ell}\rho^{\ell}\leq L_j \text{ for each } j=1,2,...,m \tag{2}$$

$$\sum_{\ell=1}^{k}\sum_{j=1}^{m}\{f_j^{\ell}\rho^{\ell}\,|\,a_i\in P^j\}\leq u_i \text{ for each } i=1,2,...,n \tag{3}$$

$$\sum_{\ell=1}^{k}\sum_{j=1}^{m} W_j^{\ell} f_j^{\ell}\leq c \tag{4}$$

Note that $\sum_{\ell=1}^{k}\sum_{j=1}^{m}\{f_j^{\ell}\rho^{\ell} \mid a_i \in P^j\}$ is the least amount of capacity needed for a_i under such a flow pattern $(f_1^1, f_1^2,..., f_1^k, f_2^1, f_2^2,..., f_2^k,..., f_m^1, f_m^2,..., f_m^k)$, and so, under the system-state vector X, $\sum_{\ell=1}^{k}\sum_{j=1}^{m}\{f_j^{\ell}\rho^{\ell} \mid a_i \in P^j\}$ does not exceed the current capacity x_i of a_i. This fact is given in the following theorem.

Theorem 1. Let $X=(x_1, x_2,..., x_n)$ be any system-state vector for which $V(X)=\mathbf{d}$. Then, the following is a necessary condition for the flow-conservation law to hold under X:

$$x_i \geq \sum_{\ell=1}^{k}\sum_{j=1}^{m}\{f_j^{\ell}\rho^{\ell} \mid a_i \in P^j\} \text{ for each } i=1,2,...,n \tag{5}$$

for any $(f_1^1, f_1^2,..., f_1^k, f_2^1, f_2^2,..., f_2^k,..., f_m^1, f_m^2,..., f_m^k)$ which is a feasible flow pattern of flow **d** under X.

Theorem 2. Let X be a (**d**;c)-MP. Then, the following is a necessary condition for the flow-conservation law to hold under X:

$$x_i = \sum_{\ell=1}^{k}\sum_{j=1}^{m}\{f_j^{\ell}\rho^{\ell} \mid a_i \in P^j\} \text{ for each } i=1,2,...,n \tag{6}$$

for any $(f_1^1, f_1^2,..., f_1^k, f_2^1, f_2^2,..., f_2^k,..., f_m^1, f_m^2,..., f_m^k)$ which is a feasible flow pattern of flow **d** under X.

Proof . By Theorem 1, $x_i \geq \sum_{\ell=1}^{k}\sum_{j=1}^{m}\{f_j^{\ell}\rho^{\ell} \mid a_i \in P^j\}$ for each $i=1,2,...,n.$

1. For each $a_i \in Z_X$, $x_i = 0$ and so (6) holds.
2. It remains to show that (6) holds for each $a_i \in N_X$. Suppose, on the contrary, that there exists an arc $a_i \in N_X$ such that $\sum_{\ell=1}^{k}\sum_{j=1}^{m}\{f_j^{\ell}\rho^{\ell} \mid a_i \in P^j\} < x_i$. Then, $\sum_{\ell=1}^{k}\sum_{j=1}^{m}\{f_j^{\ell}\rho^{\ell} \mid a_i \in P^j\} \leq x_i - 1$. In particular, $V(X-e_i)=\mathbf{d}=V(X)$, and so $a_i \notin S_X$, which contradicts to the fact that X is a (d.c)-MP. Hence, $x_i = \sum_{\ell=1}^{k}\sum_{j=1}^{m}\{f_j^{\ell}\rho^{\ell} \mid a_i \in P^j\}$ for each $a_i \in N_X$.

The vector $X=(x_1, x_2,..., x_n)$ obtained by first solving $F=(f_1^1, f_1^2,..., f_1^k, f_2^1, f_2^2,..., f_2^k,..., f_m^1, f_m^2,..., f_m^k)$ subject to constraints (1) - (4) and then transforming such $F=(f_1^1, f_1^2,..., f_1^k, f_2^1, f_2^2,..., f_2^k,..., f_m^1, f_m^2,..., f_m^k)$ to $X=(x_1, x_2,..., x_n)$ by applying the relationship in (6), will be taken as a (**d**;c)-MP candidate. To make it clearer that all (**d**;c)-MPs can be generated by the proposed method, the following theorem is necessary.

Theorem 3. Every (**d**;c)-MP is a (**d**;c)-MP candidate.

Proof. Let $X=(x_1, x_2,..., x_n)$ be any (**d**;c)-MP. By definition, we know that the maximal flow from s to t under X is **d** (i.e., $V(X)=\mathbf{d}$) and the total transmission cost is less than or equal to c. Hence, under the system-state vector *X*, there exists at least one feasible flow pattern $F=(f_1^1, f_1^2,..., f_1^k, f_2^1, f_2^2,..., f_2^k,..., f_m^1, f_m^2,..., f_m^k)$ of flow $\mathbf{d}=(d_1, d_2,..., d_k)$ such that conditions (1) - (4) are satisfied. As $X=(x_1, x_2,..., x_n)$ is a (**d**;c)-MP, we thus conclude, by Theorem 2, that $x_i = \sum_{\ell=1}^{k}\sum_{j=1}^{m}\{f_j^{\ell}\rho^{\ell} \mid a_i \in P^j\}$ for each $i=1,2,...,n$. This means that X is a (**d**;c)-MP candidate. Hence, every (**d**;c)-MP is a (**d**;c)-MP candidate.

In this article, we first find feasible solutions $F=(f_1^1, f_1^2,..., f_1^k, f_2^1, f_2^2,..., f_2^k,..., f_m^1, f_m^2,..., f_m^k)$ subject to constraints (1) - (4) by applying an implicit enumeration method (e.g., backtracking or branch-and-bound [11]) and then transform such integer-valued solutions into (**d**;c)-MP candidates $(x_1, x_2,..., x_n)$ via the relationship in (6). Each (**d**;c)-MP candidate X must be checked whether all nonzero-capacity arcs under X (i.e., $arc \in N_X$) belong to S_X . If the answer is "yes", then X is a (**d**;c)-MP. Otherwise, X is not a (**d**;c)-MP. The following two theorems play the crucial roles in checking whether a (**d**;c)-MP candidate is a (**d**;c)-MP.

Theorem 4. For each (**d**;c)-MP candidate X, there exists at least one (**d**;c)-MP Y such that $Y \le X$. In particular, X is not a (**d**;c)-MP if such a Y satisfies Y < X (where $Y \le X$ if and only if $y_i \le x_i$ for i=1, 2, ..., n and $Y < X$ if and only if $Y \le X$ and $y_i < x_i$ for at least one i).

Proof. If X is a (**d**;c)-MP, then Y must be taken as X. Suppose that X is not a (**d**;c)-MP; then there exists a nonzero-capacity arc a_i $(i.e., a_i \in N_X)$ such that $V(X-e_i)=V(X)=\mathbf{d}$. Let $X^1 = X - e_i$. Suppose that X^1 is a (**d**;c)-MP; then Y is taken as X^1. Otherwise, the same procedure may be repeated for X^1. However, this procedure will stop in finite steps, i.e., there exists an integer p such that $X^p \le X^{p-1} \le ... \le X^1 \le X$ with $V(X^p)=\mathbf{d}$ and $N_{X^p} = S_{X^p}$. The proof is thus concluded by letting $Y = X^p$.

Theorem 5. If the network is acyclic (i.e., contains no directed cycle), then each (**d**;c)-MP candidate is a (**d**;c)-MP.

Proof. Let $X=(x_1, x_2,..., x_n)$ be any (**d**;c)-MP candidate. By Theorem 4, we know that there exists a (**d**;c)-MP $Y=(y_1, y_2,..., y_n)$ such that $Y \le X$. Since $V(X-Y)=\mathbf{d}-\mathbf{d}=0$, no flow is transmitted from s to t under $X-Y=(x_1-y_1, x_2-y_2,..., x_n-y_n)$. Hence, in case $X \ne Y$, $I=\{i \mid x_i - y_i > 0\}$ is not empty and so $\{a_i \mid i \in I\}$, which is a subset of N_X , must form cycles since the flow conserves at each node (except for s and t) and there is no other sink except t (see Ford and Fulkerson [10] or Ahuja et al. [2] for more details). This means that if the network is acyclic, then $I=\phi$ and so $X = Y$, i.e., each (**d**;c)-MP candidate X is a (**d**;c)-MP .

Suppose that $X^1, X^2,..., X^q$ are total (**d**;c)-MP candidates. We can thus conclude, by Theorem 4, that X^j is a (**d**;c)-MP if $X^j \nless X^i$ for all $j=1,2,....,q$ but $j \ne i$.

IV. ALGORITHM

Suppose that all MPs, $P^1, P^2,..., P^m$, have been stipulated in advance [5-6, 24-25], the family of all (**d**;c)-MPs can then be derived by the following steps:

Step 1. For each $P^j (j=1,2,..., m)$, calculate $L_j = \min\{u_i \mid a_i \in P^j\}$ and $W_j^\ell = \sum_i \{c_i^\ell \mid a_i \in P^j\}$

Step 2. Find all feasible solutions $F=(f_1^1, f_1^2,..., f_1^k, f_2^1, f_2^2,..., f_2^k,..., f_m^1, f_m^2,..., f_m^k)$ subject to the following constraints by applying an implicit enumeration method:

(1) $\sum_{j=1}^{m} f_j^\ell = d_\ell$ for each $\ell = 1,2,..., k$

(2) $\sum_{\ell=1}^{k} f_j^\ell \rho^\ell \le L_j$ for each $j=1,2,..., m$

(3) $\sum_{\ell=1}^{k} \sum_{j=1}^{m} \{ f_j^\ell \rho^\ell \mid a_i \in P^j\} \le u_i$ for each $i=1,2,..., n$

(4) $\sum_{\ell=1}^{k} \sum_{j=1}^{m} W_j^\ell f_j^\ell \le c$

where f_j^ℓ is a nonnegative integer for $j = 1,2,..., m$ and $\ell = 1,2,..., k$.

Step 3. Transform the solutions $(f_1^1, f_1^2,..., f_1^k, f_2^1, f_2^2,..., f_2^k,..., f_m^1, f_m^2,..., f_m^k)$ into (**d**;c)-MP candidates $X = (x_1, x_2,..., x_n)$ via $x_i = \sum_{\ell=1}^{k}\sum_{j=1}^{m}\{f_j^\ell \rho^\ell \mid a_i \in P^j\}$ for $i = 1,2,..., n$.

Step 4. Check each candidate X one at a time whether it is a (**d**;c)-MP:

(A) If the network is acyclic, then each candidate is a (**d**;c)-MP.

(B) If the network is cyclic, and suppose $\{X^1, X^2,..., X^q\}$ is the family of all such (**d**;c)-MP candidates, then X^i is a (**d**;c)-MP if $X^j \not< X^i$ for all $j = 1,2,..., q$ but $j \neq i$.

V. EXAMPLES

The following two examples are used to illustrate the proposed algorithm:

Example 1.

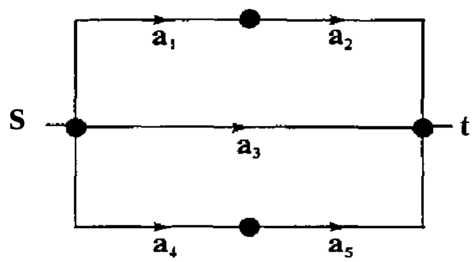

FIG. 1. A SERIES-PARALLEL NETWORK.

Consider the network in Fig. 1. It is known that $U = (u_1, u_2, u_3, u_4, u_5) = (1,2,2,2,1)$, $C = (2,3,2,3,5,6,2,3,2,3)$, $\boldsymbol{\rho} = (\rho_1, \rho_2) = (1,2)$, and there exists three MPs; $P^1 = \{a_1, a_2\}, P^2 = \{a_3\}, P^3 = \{a_4, a_5\}$. Given $\mathbf{d} = (1,1)$ and $c = 12$, the family of (**d**;c)-MPs is derived as follows:

Step 1. $L_1 = \min\{1,2\} = 1,$ $L_2 = \min\{2\} = 2,$ $L_3 = \min\{2,1\} = 1,$ $W_1^1 = 2 + 2 = 4,$ $W_1^2 = 3 + 3 = 6,$ $W_2^1 = 5,$ $W_2^2 = 6,$ $W_3^1 = 2 + 2 = 4,$ $W_3^2 = 3 + 3 = 6.$

Step 2. Find all feasible solutions $(f_1^1, f_1^2, f_2^1, f_2^2, f_3^1, f_3^2)$ subject to the following constraints by applying an implicit enumeration method:

$$\begin{cases} f_1^1 + f_2^1 + f_3^1 = 1 \\ f_1^2 + f_2^2 + f_3^2 = 1 \end{cases}$$

$$\begin{cases} f_1^1 \times 1 + f_1^2 \times 2 \le 1 \\ f_2^1 \times 1 + f_2^2 \times 2 \le 2 \\ f_3^1 \times 1 + f_3^2 \times 2 \le 1 \end{cases}$$

$$\begin{cases} f_1^1 \times 1 + f_1^2 \times 2 \le 1 \\ f_1^1 \times 1 + f_1^2 \times 2 \le 2 \\ f_2^1 \times 1 + f_2^2 \times 2 \le 2 \\ f_3^1 \times 1 + f_3^2 \times 2 \le 2 \\ f_3^1 \times 1 + f_3^2 \times 2 \le 1 \end{cases}$$

$$4f_1^1 + 6f_1^2 + 5f_2^1 + 6f_2^2 + 4f_3^1 + 6f_3^2 \le 12$$

where f_j^ℓ is a nonnegative integer for $j = 1,2,3$ and $\ell = 1,2.$

Total feasible solutions are $F^1 = (1,0,0,1,0,0)$ and $F^2 = (0,0,0,1,1,0).$

Step 3. Transform such feasible solutions into (**d**;c)-MP candidates $X=(x_1,x_2,x_3,x_4,x_5)$ via $x_i=\sum_{\ell=1}^{3}\sum_{j}\{f_j^\ell \rho^\ell \mid a_i \in P^j\}$ for $i=1,2,\ldots,5$. Then $X^1=(1,1,2,0,0)$ and $X^2=(0,0,2,1,1)$ are total (**d**;c)-MP candidates.

Step 4. The network is acyclic, and $\{X^1,X^2\}$ is the family of all (**d**;c)-MP candidates. Since $X^i \nless X^j$, $X^1=(1,1,2,0,0)$ and $X^2=(0,0,2,1,1)$ are total (**d**;c)-MP

Example 2.

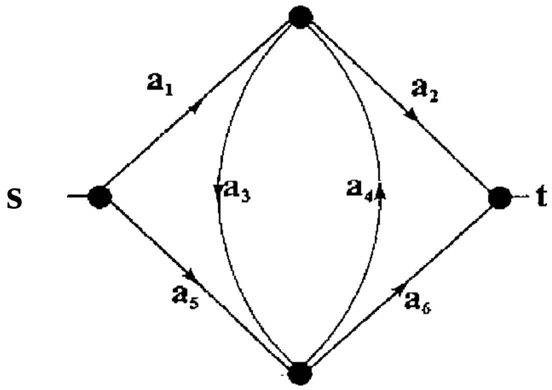

FIG. 2. A BRIDGE NETWORK.

TABLE 1
PROBABILITY DISTRIBUTIONS OF ARC CAPACITIES IN EXAMPLE 2

Arc	Capacity	Probability	Arc	Capacity	Probability
a_1	3	0.60	a_4	1	0.90
	2	0.25		0	0.10
	1	0.10	a_5	2	0.80
	0	0.05		1	0.15
a_2	2	0.70		0	0.05
	1	0.20	a_6	3	0.65
	0	0.10		2	0.20
a_3	1	0.90		1	0.10
	0	0.10		0	0.05

TABLE 2
UNIT TRANSMISSION COST ON EACH ARC IN EXAMPLE 2

Arc	Commodity	Cost	Arc	Commodity	Cost
a_1	1	2	a_4	1	1
	2	2		2	1
	3	3		3	2
a_2	1	2	a_5	1	2
	2	3		2	3
	3	3		3	3
a_3	1	1	a_6	1	2
	2	1		2	2
	3	2		3	3

Consider the network in Fig. 2. It is known that $U=(u_1,u_2,u_3,u_4,u_5,u_6)=(3,2,1,1,2,3)$, $\boldsymbol{\rho}=(\rho_1,\rho_2,\rho_3)=(1,2,1)$, and there exists four MPs; $P^1=\{a_1,a_2\}, P^2=\{a_1,a_3,a_6\}, P^3=\{a_2,a_4,a_5\}, P^4=\{a_5,a_6\}$. Given $\mathbf{d}=(1,1,1)$ and $c=16$, the family of (**d**;c)-MPs is derived as follows:

Step 1. $L_1 = \min\{3,2\} = 2,\ L_2 = \min\{3,1,3\} = 1,\ L_3 = \min\{2,1,2\} = 1,\ L_4 = \min\{2,3\} = 2,$

$W_1^1 = 2+2 = 4,\ W_1^2 = 2+3 = 5,\ W_1^3 = 3+3 = 6,\ W_2^1 = 2+1+2 = 5,\ W_2^2 = 2+1+2 = 5,\ W_2^3 = 3+2+3 = 8,$

$W_3^1 = 2+1+2 = 5,\ W_3^2 = 3+1+3 = 7,\ W_3^3 = 3+2+3 = 8, W_4^1 = 2+2 = 4,\ W_4^2 = 3+2 = 5,$

and $W_4^3 = 3+3 = 6.$

Step 2. Find all feasible solutions $(f_1^1, f_1^2, f_1^3, f_2^1, f_2^2, f_2^3, f_3^1, f_3^2, f_3^3, f_4^1, f_4^2, f_4^3)$ subject to the following constraints by applying an implicit enumeration method:

$$\begin{cases} f_1^1 + f_2^1 + f_3^1 + f_4^1 = 1 \\ f_1^2 + f_2^2 + f_3^2 + f_4^2 = 1 \\ f_1^3 + f_2^3 + f_3^3 + f_4^3 = 1 \end{cases}$$

$$\begin{cases} f_1^1 \times 1 + f_1^2 \times 2 + f_1^3 \times 1 \le 2 \\ f_2^1 \times 1 + f_2^2 \times 2 + f_2^3 \times 1 \le 1 \\ f_3^1 \times 1 + f_3^2 \times 2 + f_3^3 \times 1 \le 1 \\ f_4^1 \times 1 + f_4^2 \times 2 + f_4^3 \times 1 \le 2 \end{cases}$$

$$\begin{cases} f_1^1 \times 1 + f_2^1 \times 1 + f_1^2 \times 2 + f_2^2 \times 2 + f_1^3 \times 1 + f_2^3 \times 1 \le 3 \\ f_1^1 \times 1 + f_3^1 \times 1 + f_1^2 \times 2 + f_3^2 \times 2 + f_1^3 \times 1 + f_3^3 \times 1 \le 2 \\ f_2^1 \times 1 + f_2^2 \times 2 + f_2^3 \times 1 \le 1 \\ f_3^1 \times 1 + f_3^2 \times 2 + f_3^3 \times 1 \le 1 \\ f_3^1 \times 1 + f_4^1 \times 1 + f_3^2 \times 2 + f_4^2 \times 2 + f_3^3 \times 1 + f_4^3 \times 1 \le 2 \\ f_2^1 \times 1 + f_4^1 \times 1 + f_2^2 \times 2 + f_4^2 \times 2 + f_2^3 \times 1 + f_4^3 \times 1 \le 3 \end{cases}$$

$$4f_1^1 + 5f_1^2 + 6f_1^3 + 5f_2^1 + 5f_2^2 + 8f_2^3 + 5f_3^1 + 7f_3^2 + 8f_3^3 + 4f_4^1 + 5f_4^2 + 6f_4^3 \le 16$$

where f_j^ℓ is a nonnegative integer for $j = 1,2,3,4$ and $\ell = 1,2,3.$

Total feasible solutions are $F^1 = (0,0,1,1,0,0,0,0,0,0,1,0)$, $F^2 = (0,1,0,0,0,0,0,0,0,1,0,1)$, $F^3 = (0,1,0,1,0,0,0,0,0,0,0,1)$, and $F^4 = (1,0,1,0,0,0,0,0,0,0,1,0)$.

Step 3. Transform such feasible solutions into (**d**;c)-MP candidates $X = (x_1, x_2, x_3, x_4, x_5, x_6)$ via $x_i = \sum_{\ell=1}^{3} \sum_j \{ f_j^\ell \rho^\ell \mid a_i \in P^j \}$ for $i = 1,2,...,6$. Then $X^1 = (2,1,1,0,2,3)$, $X^2 = (2,2,0,0,2,2)$, and $X^3 = (3,2,1,0,1,2)$ are total (**d**;c)-MP candidates.

Step 4. The network is cyclic, and $\{X^1, X^2, X^3\}$ is the family of all (**d**;c)-MP candidates. Since $X^i \not< X^j$, every (**d**;c)-MP candidate is a (**d**;c)-MP. The result is listed in Table 3.

TABLE 3
LIST OF ALL (D;C)-MPS IN EXAMPLE 2

(d;c)-MP candidate	(d;c)-MP?
$X^1 = (2,1,1,0,2,3)$	Yes
$X^2 = (2,2,0,0,2,2)$	Yes
$X^3 = (3,2,1,0,1,2)$	Yes

VI. RELIABILITY EVALUATION

If $Y^1, Y^2, ..., Y^{m_{(\mathbf{d};c)}}$ are the collection of all (**d**;c)-MPs, then the system reliability for level (**d**;c) is defined as $R_{(\mathbf{d};\mathbf{c})} = \Pr\{\cup_{i=1}^{m_{(\mathbf{d};c)}} \{X \mid X \geq Y^i\}\}$. To compute it, several methods such as inclusion-exclusion [8, 12], disjoint subset [13], and state-space decomposition [4] are available. Here we apply the state-space decomposition method to Example 2 and obtain that $R_{(\mathbf{d};\mathbf{c})} = \Pr\{\cup_{i=1}^{m_{(\mathbf{d};c)}} \{X \mid X \geq Y^i\}\} = 0.53235$ for demand level $\mathbf{d} = (1,1,1)$ and $c = 16$.

VII. CONCLUSION

Given all MPs that are stipulated in advance, the proposed method can generate all (**d**;c)-MPs of a multicommodity limited-flow network under budget constraints for each level (**d**;c). The system reliability, i.e., the probability that k different types of commodity can be transmitted from the source node s to the sink node t in the way that the demand level $\mathbf{d} = (d_1, d_2, ..., d_k)$ is satisfied and the total transmission cost is less than or equal to c, can then be computed in terms of these (**d**;c)-MPs. This algorithm can also apply to the limited-flow network with single commodity. Hence, earlier algorithm [18] is shown to be a special case of this new one.

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