Synthesis of Discrete Steady-State Error Free Modal State Controller Based on Predefined Pole Placement Area and Measurable State Variables

Mariela Alexandrova

Department of Automation, Technical University of Varna, Bulgaria
Email: m_alexandrova@tu-varna.bg

Abstract— *The entire case study aims to present a procedure for synthesis of discrete steady-state error free modal state controller, based on predefined pole placement area and measurable state variables, suggesting a solution for the so called pole placement problem (PPP) (known also as pole assignment problem - PAP). It will introduce a research on how the suggested solution will result in modeling of dual-mass DC electromechanical plant.*

Keywords*— *controller synthesis, discrete modal state controller, modal control, state feedback, state space, state space model, pole assignment, pole placement

I. INTRODUCTION

One of the most frequently used approaches to dynamic response analysis of linear time-invariant systems modeling, is so called modal control, which is related to the assignment of the closed-loop poles to an arbitrary predefined set of points in the complex plane. For continuous-time systems the pole placement could be assigned to predefined values or into areas which should be distributed in the left half plane of the complex plane. In terms of discrete-time systems areas desired must be within the unit circle in the z-plane. Regardless of the method chosen for the pole placement there are two different approaches depending on the information used in control processing: output feedback pole placement and state feedback pole placement.

The entire case study will develop the second approach state feedback pole placement for which are known two possible solutions – all state space variables are measurable and known and the second option is related to information only partially available for the state space variables which is a little bit more complex task in terms of analysis and realization of modal control methods. Thus it will be a matter of further investigations.

II. FORMULATION OF THE METHOD USED, PROBLEM STATEMENT AND PRELIMINARIES

The modal control will be based on the method related to the second approach of the pole placement procedure – predefined pole placement area which in this case in particular will be chosen to be elliptical area of distribution in the unit circle.

The object taken into consideration for modal control is a dual-mass DC electromechanical plant [2,3]. The block diagram of this is presented on Fig.1

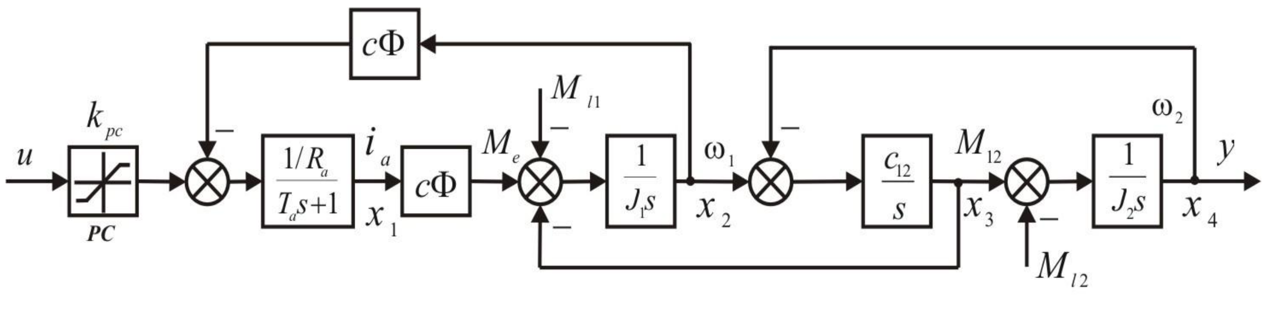

FIGURE 1. BLOCK-DIAGRAM OF THE PLANT UNDER CONSIDERATION

2.1 Continuous state space model composition

Linear continuous-time system state space model taken into consideration is given as follows:

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{A}x + \boldsymbol{b}u + \boldsymbol{v} \\ y &= \boldsymbol{c}^{\boldsymbol{T}}x + \boldsymbol{d}u\end{aligned} \tag{1}$$

where x is an *n*-dimensional state vector, *u* is an *r*-dimensional input vector, and *y* is an *m*-dimensional output vector, ***A***, ***b***, ***c***, ***d*** are the system matrices of appropriate sizes and ***v*** is the disturbance input matrix.

Based on the practical point of view a set of state variables is selected, corresponding to real physical parameters, as listed below: x_1 - armature current i_a; x_2 - rotor speed ω_1; x_3 – engine elastic torque M_{12} and x_4 – working/actuating machine speed ω_2 (plant output). According to the block diagram shown on Fig.1 the state space variables could be presented by the following equations:

$$\begin{cases} \dot{x}_1 = \frac{1}{T_a R_a}\left(k_{pc}u - c\Phi x_2\right) - \frac{1}{T_a}x_1 \\ \dot{x}_2 = \frac{1}{J_1}\left(c\Phi x_1 - x_3 - M_{l1}\right) \\ \dot{x}_3 = c_{12}\left(x_2 - x_4\right) \\ \dot{x}_4 = \frac{1}{J_2}\left(x_3 - M_{l2}\right) \end{cases} \quad (2)$$

Therefore matrices ***A***, ***b***, ***c***, ***d*** and ***v*** of equation (1) are:

$$\boldsymbol{A} = \begin{bmatrix} -\frac{1}{T_a} & -\frac{c\Phi}{T_a R_a} & 0 & 0 \\ \frac{c\Phi}{J_1} & 0 & -\frac{1}{J_1} & 0 \\ 0 & c_{12} & 0 & -c_{12} \\ 0 & 0 & \frac{1}{J_2} & 0 \end{bmatrix}; \boldsymbol{b} = \begin{bmatrix} \frac{k_{pc}}{R_a T_a} \\ 0 \\ 0 \\ 0 \end{bmatrix}; \boldsymbol{c}^{\boldsymbol{T}} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}; \boldsymbol{d} = 0; \boldsymbol{v} = \begin{bmatrix} 0 & 0 \\ -\frac{1}{J_1} & 0 \\ 0 & 0 \\ 0 & -\frac{1}{J_2} \end{bmatrix} \begin{bmatrix} M_{l1} \\ M_{l2} \end{bmatrix} \quad (3)$$

2.2 Discrete state space model derivation

Similar to the continuous system the discrete one could be also presented through a model in state space using the form given below:

$$\begin{aligned} x(k+1) &= \boldsymbol{A}^* x(k) + \boldsymbol{b}^* u(k) + \boldsymbol{v}^*(k) \\ y(k) &= \boldsymbol{c}^{*\boldsymbol{T}} x(k) + \boldsymbol{d}^* u(k) \end{aligned} \quad (4)$$

For the system, given with equation (4), we can consider the following pole assignment problem (PAP) solution: determination of the feedback matrix $\boldsymbol{f}^*$ using the same method for the closed-loop state matrix $\boldsymbol{A}^* - \boldsymbol{b}^*\boldsymbol{f}^*$ has its eigenvalues distributed inside the desired area $\mathbb{C}_\varepsilon$ (into *z* plane). If we assume that $\mathbb{C}_\varepsilon$ is an elliptical area symmetric according the real axis, then this will guarantee that the elements of matrix $\boldsymbol{f}^*$ are real numbers. The desired area is given by the following equation:

$$z(r_1, r_2, \mu, z_{new}) = \mu + r_1 cos(\omega T_q) + j r_2 sin(\omega T_q) = \frac{1}{2z_{new}}\left[(r_1 + r_2)z_{new}^2 + 2\mu z_{new} + (r_1 - r_2)\right] \quad (5)$$

where: r_1–major semi-axis; r_2 - minor semi-axis; μ - distance from the center to the origin; T_q – discretization period.

Immediate response is required for the closed-loop system even if input disturbance occur. Placing the poles in $\mathbb{C}_\varepsilon$, implementing appropriate state feedback leads to more or less overdamped time responses of the closed-loop system. The result will be settling time lower than the plant and steady-state error free.

III. RESULTS

Algorithm for identifying a desired feedback matrix $\boldsymbol{f}^*$ - to satisfy the requirements all the closed-loop poles to be distributed on and into the defined area given by equation (5)

Step 1: Find state matrix of the discrete open-loop system in three diagonal band form $\boldsymbol{A}^* \xrightarrow{eig(A^*)=eig(A_T^*)} \boldsymbol{A}_T^*$.

Step 2: Determination of 'working' matrix $\boldsymbol{A}_{Tw}^{*}$ through correction of the elements of $\boldsymbol{A}_{T}^{*}$: $\boldsymbol{A}_{T}^{*} \xrightarrow{correction} \boldsymbol{A}_{Tw}^{*}$.

Step 3: Determination of eigenvalues of $\boldsymbol{A}_{Topt}^{*}$: $\lambda^{*'} = eig\left(\boldsymbol{A}_{Topt}^{*}\right)$.

Step 4: Determination of eigenvalues λ_{newi}^{*} set on and into the unit circle: $\lambda_{newi}^{*} = \frac{1+\lambda_i^{*'}}{1-\lambda_i^{*'}}, i = \overline{1,n}$.

Step 5: Ensure the desire eigenvalues to be set on and into the defined area $\mathbb{C}_{\varepsilon}$:

$$\lambda_{di}^{*} = \frac{1}{2\lambda_{newi}^{*}}\left[(r_1 + r_2)\lambda_{newi}^{*2} + 2\mu\lambda_{newi}^{*} + (r_1 - r_2)\right], i = \overline{1,n}.$$

Step 6: Determination of desired polynomial $h_d^{*}(z) = poly((\lambda_d^{*})^{T})$.

In the algorithm shown above $\boldsymbol{A}_{T}^{*}$ is three-diagonal band matrix found through the elements of the state matrix $\boldsymbol{A}^{*}$ determined in accordance with the requirement for equality of the eigenvalues (*Step 1*). Conclusion about system stability could be done based on the signs and values of its elements [1,4]. Correction of values of the elements is performed in a manner that eigenvalues of $\boldsymbol{A}_{T}^{*}$ ($eig(\boldsymbol{A}_{T}^{*})$) will be positioned on and in the left side of imaginary axis (*Step 2*). There is a number of options available for use however the procedure could be reduced only to meet the requirement for positive signs of all the sub-diagonal elements of $\boldsymbol{A}_{T}^{*}$ and also to have one complex conjugated pair poles laying on $\mathbb{C}_{\varepsilon}$ border. Fulfillment of this requirement corresponds to a zero element in the matrix row before the last one. The presence of one real pole on the region border corresponds to existence of one zero element on the last matrix row. Desired characteristic polynomial $h_d^{*}(z)$ (determined in *Step 6*) contains information about the required feedback matrix $\boldsymbol{f}^{*}$ which could be obtained using the following equation:

$$f_i^{*} = a_{di}^{*} - a_i^{*}, i = \overline{1,n} \tag{6}$$

where a_{di}^{*} are desired polynomial coefficients and a_i^{*} are initial polynomial coefficients.

For the closed-loop system with modal controller the prescribed pole position is ensured initially by input $u_{set} = u - \boldsymbol{f}^{*}\boldsymbol{x}$. Even if the algorithm given above provides the desired pole placement there is still an essential disadvantage – resulting closed-loop system is still not steady state error free according to the output ω_2. There is a general solution of this problem and it could be solved if the input signal is determined by the following dependency: $k_i^{*}x_i(k) - \boldsymbol{f}^{*}\boldsymbol{x}(k)$ where $x_i(k+1) = k_0^{*}g(k) - \boldsymbol{c}^{*T}\boldsymbol{x}(k)$. Once it is implemented the new state space variable is an output of discrete integrator with coefficient k_i^{*}. The result will be equivalent discrete five state space variables closed-loop system given by the below equation:

$$x_{equiv}(k+1) = \boldsymbol{A}_{equiv}^{*}x_{equiv}(k) + \boldsymbol{G}^{*}g(k) + \boldsymbol{M}^{*}\boldsymbol{M}_l(k) \tag{7}$$

where $\boldsymbol{x}_{equiv} = \begin{bmatrix} x_i \\ \boldsymbol{x} \end{bmatrix}$, $\boldsymbol{A}_{equiv}^{*} = \boldsymbol{A}_{5plant}^{*} + \boldsymbol{b}_{5plant}^{*}\boldsymbol{k}_{fb}^{*}$, $\boldsymbol{G}^{*} = [T_q k_0^{*} \quad 0 \quad 0 \quad 0 \quad 0]^{T}$, $\boldsymbol{M}_l(k) = \begin{bmatrix} M_{l1}(k) \\ M_{l2}(k) \end{bmatrix}$,

$\boldsymbol{k}_{fb}^{*} = [k_i^{*} \quad -f_1^{*} \quad -f_2^{*} \quad -f_3^{*} \quad -f_4^{*}]$ - matrix of unknown state feedback coefficients and integrator coefficient k_i^{*}.

IV. SIMULATION

Dual-mass DC electromechanical plant shown on Fig.1 is operative through DC motor with the following parameters [2,3]:

$U_{nom} = 220V;\ P_{nom} = 0{,}3kW;\ n_{nom} = 1000\ tr/min;\ I_{anom} = 2A;\ u \to 0 \div 10V;$
$R_a = 20{,}8828\Omega;\ T_a = 0{,}0126s;\ c\Phi = 1{,}702Vs;\ c_{12} = 1{,}702Nm;$
$k_{pc} = 22;\ J_1 = 0{,}042kgm^2;\ J_2 = 0{,}021kg\,m^2;\ M_{l1} = 0{,}3404Nm;\ M_{l2} = 3{,}0636Nm$

In accordance with equation (3) for continuous-time state space system model we calculate:

$$\boldsymbol{A}=\begin{bmatrix}-\frac{1}{T_a} & -\frac{c\Phi}{T_aR_a} & 0 & 0\\ \frac{c\Phi}{J_1} & 0 & -\frac{1}{J_1} & 0\\ 0 & c_{12} & 0 & -c_{12}\\ 0 & 0 & \frac{1}{J_2} & 0\end{bmatrix}=\begin{bmatrix}-79{,}3651 & -6{,}468451 & 0 & 0\\ 40{,}52381 & 0 & -23{,}8095 & 0\\ 0 & 1{,}702 & 0 & -1{,}702\\ 0 & 0 & 47{,}619 & 0\end{bmatrix},\ \boldsymbol{b}=\begin{bmatrix}\frac{k_{pc}}{R_aT_a}\\ 0\\ 0\\ 0\end{bmatrix}=\begin{bmatrix}83{,}611\\ 0\\ 0\\ 0\end{bmatrix}$$

$$\boldsymbol{c}^{\boldsymbol{T}}=[0\quad 0\quad 0\quad 1],\boldsymbol{d}=0,\boldsymbol{v}=\begin{bmatrix}0 & 0\\ -\frac{1}{J_1} & 0\\ 0 & 0\\ 0 & -\frac{1}{J_2}\end{bmatrix}\begin{bmatrix}M_{r1}\\ M_{r2}\end{bmatrix}=\begin{bmatrix}0 & 0\\ -23{,}8095 & 0\\ 0 & 0\\ 0 & -47{,}619\end{bmatrix}\begin{bmatrix}0{,}3403\\ 3{,}0636\end{bmatrix}$$

The calculations are made in MATLAB environment and the discrete model of the system is formulated based on the continuous model of equation (1), considering (2) and (3).

Discrete-time state space model could be obtained using MATLAB function for continuous to discrete model transition – *c2d*(.). Discretization period is defined $T_q = 0{,}1s$ which is chosen based on requirement $T_q \leq (0{,}05 \div 0{,}1)t_c$. In this case t_c (forecasted control time) is set to be equal to the expectation for the continuous-time closed-loop system [3]. Considering these assumptions for the discrete-time state-space model (4) is calculated:

$$\boldsymbol{A}^{*}=\begin{bmatrix}-0{,}0291 & -0{,}05289 & 0{,}127 & -0{,}01104\\ 0{,}3313 & 0{,}5864 & -1{,}672 & 0{,}1689\\ 0{,}05688 & 0{,}1195 & 0{,}4648 & -0{,}1383\\ 0{,}1383 & 0{,}3378 & 3{,}869 & 0{,}6337\end{bmatrix},\boldsymbol{b}^{*}=\begin{bmatrix}0{,}0376\\ 0{,}1438\\ 0{,}0110\\ 0{,}0168\end{bmatrix},\boldsymbol{c}^{*\boldsymbol{T}}=[0\quad 0\quad 0\quad 1];\ \boldsymbol{d}^{*}=0$$

$$\boldsymbol{v}^{*}(k)=\begin{bmatrix}0{,}1438 & 0{,}01675\\ -1{,}959 & -0{,}2868\\ -0{,}1689 & 0{,}3663\\ -0{,}2868 & -4{,}156\end{bmatrix}\begin{bmatrix}M_{l1}(k)\\ M_{l2}(k)\end{bmatrix}$$

As per the discrete-time state-space model the plant system is fully controllable since the matrix of controllability is identified as one of full rank, i.e. $rank\left(\boldsymbol{Q}_y^{*}\right)=4$.

The input signal may vary within the range $u \rightarrow 1 \div 10V$ which corresponds to variations of the output voltage of the power converter (*PC*) within the range $0 \div 220V$. The current experiment is performed using an input signal $u = 9V$ for all experimental data shown on Fig.2. The disturbance $M_{l2} = 3{,}0636\ Nm$ is applied at the time when the plant processes are settled already (at the 8th second of the experiment start time).

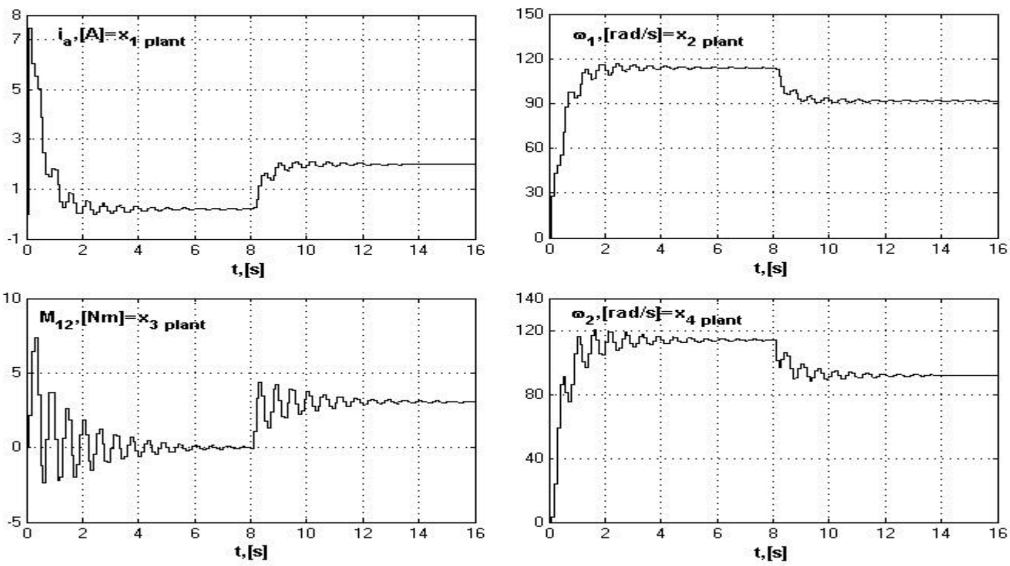

FIGURE 2. START PROCESSES OF THE PLANT

Following the equation (7) five state space variables plant matrices $\boldsymbol{A}^{*}_{5plant}$ and $\boldsymbol{b}^{*}_{5plant}$ and the closed-loop matrices $\boldsymbol{A}^{*}_{equiv}$, $\boldsymbol{G}^{*}$ and $\boldsymbol{M}^{*}\boldsymbol{M}_{l}(k)$ are as follows:

$$\boldsymbol{A}^{*}_{5plant}=\begin{bmatrix}1 & 0 & 0 & 0 & -T_q\\ 0 & -0{,}0291 & -0{,}0529 & 0{,}1270 & -0{,}0110\\ 0 & 0{,}3313 & 0{,}5864 & -1{,}6720 & 0{,}1689\\ 0 & 0{,}0569 & 0{,}1195 & 0{,}4648 & -0{,}1383\\ 0 & 0{,}1383 & 0{,}3378 & 3{,}8690 & 0{,}6337\end{bmatrix},\ \boldsymbol{b}^{*}_{5plant}=\begin{bmatrix}0\\ 0{,}0376k_{pc}\\ 0{,}1438k_{pc}\\ 0{,}0110k_{pc}\\ 0{,}0168k_{pc}\end{bmatrix}$$

$$\boldsymbol{A}^{*}_{equiv}=\begin{bmatrix}1 & 0 & 0 & 0 & -T_q\\ 0{,}0376\,k_{pc}k_i^* & -0{,}0291-0{,}8264f_1^* & -0{,}0529-0{,}8264f_2^* & 0{,}1270-0{,}8264f_3^* & -0{,}0110-0{,}8264f_4^*\\ 0{,}1438\,k_{pc}k_i^* & 0{,}3313-3{,}1630f_1^* & 0{,}5864-3{,}163f_2^* & -1{,}6720-3{,}1630f_3^* & 0{,}1689-3{,}1630f_4^*\\ 0{,}0110\,k_{pc}k_i^* & 0{,}0569-0{,}2429f_1^* & 0{,}1195-0{,}2429f_2^* & 0{,}4648-0{,}2429f_3^* & -0{,}1383-0{,}2429f_4^*\\ 0{,}0168\,k_{pc}k_i^* & 0{,}1383-0{,}3686f_1^* & 0{,}3378-0{,}3686f_2^* & 3{,}8690-0{,}3686f_3^* & 0{,}6337-0{,}3686f_4^*\end{bmatrix}$$

$$\boldsymbol{G}^{*}=[1{,}021\quad 0\quad 0\quad 0\quad 0]^{T},\qquad \boldsymbol{M}^{*}\boldsymbol{M}_{l}(k)=\begin{bmatrix}0 & 0\\ 0{,}1438 & 0{,}01675\\ -1{,}959 & -0{,}2868\\ -0{,}1689 & 0{,}3663\\ -0{,}2868 & -4{,}156\end{bmatrix}\begin{bmatrix}0{,}3404\\ 3{,}0636\end{bmatrix}$$

The parameters of area $\mathbb{C}_e$ are assumed as: major semi-axis $r_1 = 0{,}3$, minor semi-axis $r_2 = 0{,}15$, distance from the center of the elliptic area to the origin $\mu = 0{,}5$.

For isolating the elements of the state feedback matrix $\boldsymbol{k}^{*}_{fb}$ is used algorithm shown above. The defined algorithm is applied to $\boldsymbol{A}^{*}_{5plant}$ in order to obtain the coefficients a^{*}_{di}. Coefficients a^{*}_{i} are determined by $\boldsymbol{A}^{*}_{equiv}$. As a result the state feedback matrix is calculated as follows: $\boldsymbol{k}^{*}_{fb} = [0{,}5088\quad 0{,}1267\quad 0{,}26\quad 0{,}4669\quad -0{,}0587]$.

The transient processes for the state-space variables are shown on Fig.3 for the plant and for the closed-loop system: input signals $u = 9V$ and $g = 9V$. Disturbance $M_{l2} = 3{,}0636Nm$ is applied at the 8th second.

For alteration range unification of output variables x_{4plant} and $x_{4closed-loop}$ is implemented scale factor with coefficient $k_0^* = 10{,}21$.

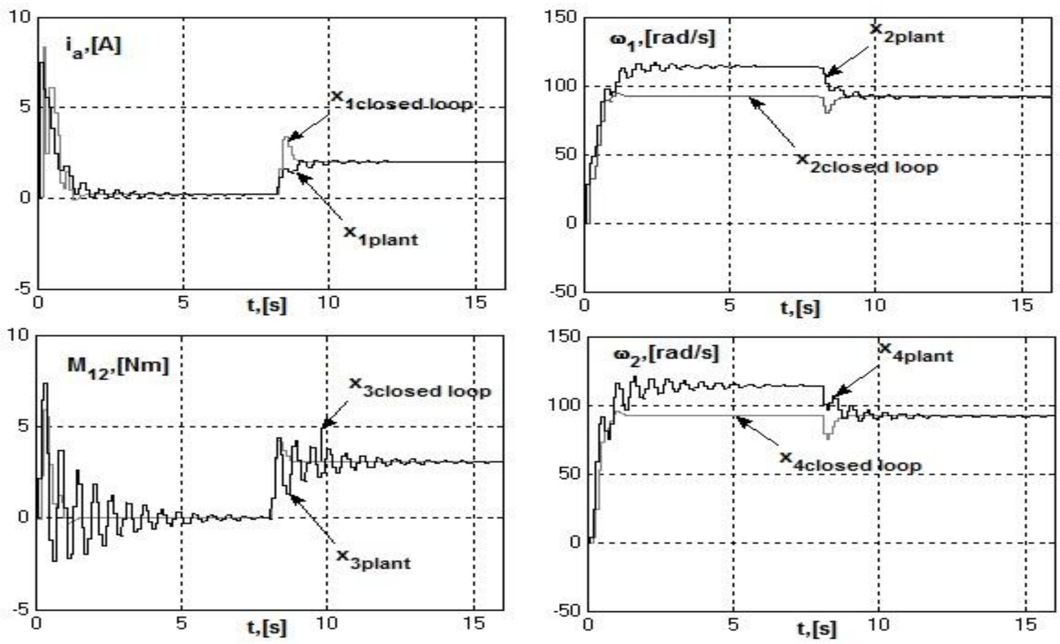

FIGURE 3. PLANT AND CLOSE-LOOP STATE VARIABLE START PROCESSES

The block diagram of the closed loop systems is shown on Fig.4, which is transformed in Simulink/MATLAB modeling utilities. The resulted model is used for simulation of the working processes in the system to support the analysis and verify the ability of the modal controller to work in combination with the rest system components.

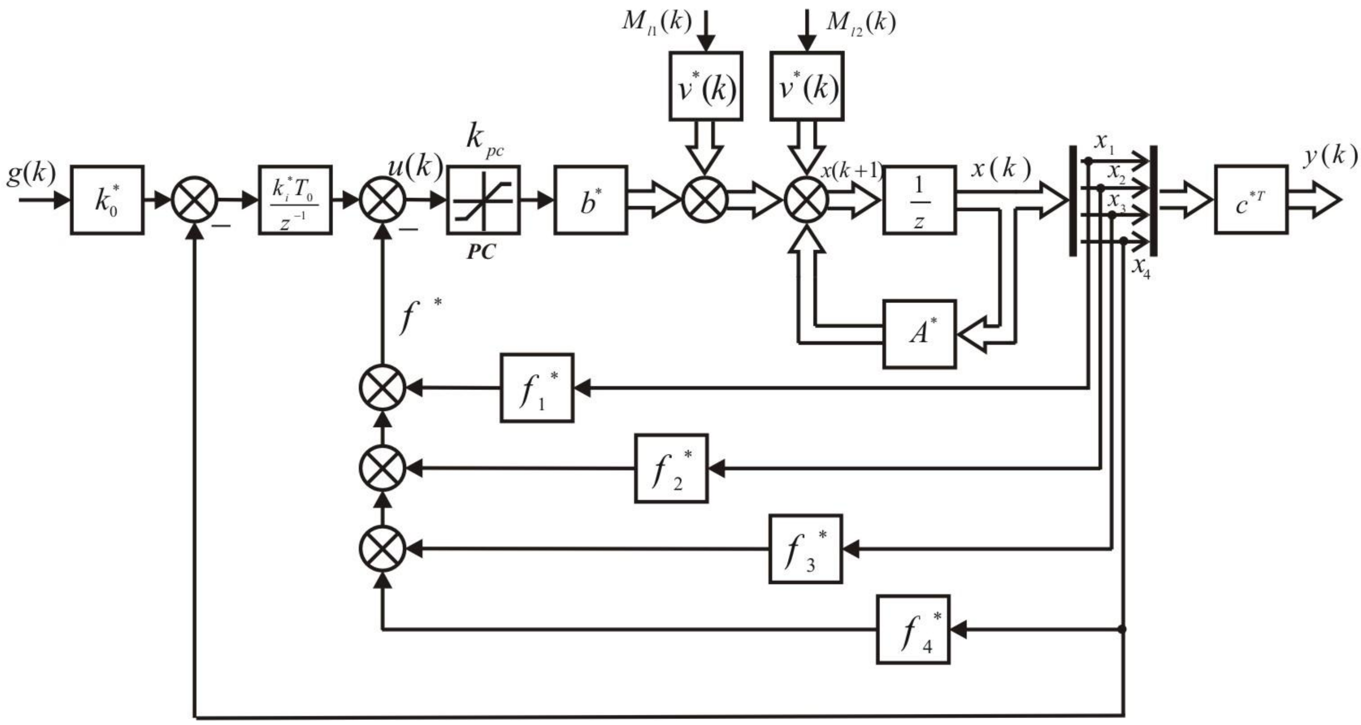

FIGURE 4. BLOCK DIAGRAM OF THE CLOSE-LOOP SYSTEM

V. GENERAL CONCLUSIONS

- The method of pole assignment is adequately and successfully used for synthesis of discrete model in MATLAB environment of modal state controller.
- The algorithm suggested is fully applicable for modeling and experimental research of the parameters of discrete modal steady-state error free state controller with measurable state variables.

VI. FUTURE DEVELOPMENTS

The entire paper concerns only one of the methods possible for pole assignments thus future research will be developed involving the case where not all of the state-space variables are measurable and known using observer to create a special response (feedback) to compensate (reconstruct) the variables for which we do not have complete information or are hardly adaptable by some reason to the algorithm used in the entire research.

REFERENCES

[1] M.I. Alexandrova, N.N. Nikolov, V.S. Dimitrov, Modal Controllers Synthesis Based On Usage Of Stability Criterions, International Conference Automatics, Control and Information Technologies, Proceedings, p.81-p.85, Proceedings, ISSN:1311-896X, 22-24 September 2010, Technical University of Varna, Varna, Bulgaria.

[2] M.R. Mikhov, T.T.Georgiev, Optimal Modal Control of Dual-Mass Electromechanical Systems, Proceedings EDS'08 IMAPS CS International Conferecne, pp.47-55, Invited Papers Brno, Czech Republic, September 10-11, 2008.

[3] N.N. Nikolov, V.S. Dimitrov, M.I. Alexandrova, I. Penev, Algorithm for Modal Control of Dual-mass Electromechanical System, ICEST 2011, Proceeding of Papers, Nish, Serbia, 2011.

[4] V.S. Dimitrov, M.I. Uzuntoneva (Alexandrova), Matrix Stability Criterion Based On A Canonical Representation**,** International Conference Automatics and Informatics'06, Proceedings, p.311-314, ISBN-10: 954-9641-48-1/ ISBN-13: 978-954-9641-48-6, 3-6 October 2006, Sofia, Bulgaria.

ADDITIONAL READINGS

[1] A. Varga, Multishift Algorithm For Pole Assignment of Single-Input Systems, Proc. ECC'95, Rome, Italy, 1995, vol.4, pp.3348-3352.
[2] B. Porter, R. Crossley, Modal Control – Theory and applications, Univeristy of Salford, Taylor&Francis LTD, London, 1972.
[3] J.D. Simon, S.K. Mitter, A Theory of Modal Control, Information and Control, vol.13, 1968, pp.316-353.
[4] P.Hr. Petkov, N.D. Christov and M.M. Konstantinov, A Computational Algorithm for Pole Assignment Of Linear Single-Input Systems, IEEE Transactions on Automatic Control, vol.AC-29, Nov. 1984, pp.1045-1048.
[5] W.J. Palm III, Modeling, Analysis, and Control of Dynamic Systems, Second Edition, John Wiley&Sons, Inc., University of Rhode Island, New York, USA, 2001.
[6] W. Mehrmann, H. Xu, An Analysis of Pole Placement Problem.I.The Single-Input Case, Electronic Transactions on Numerical Analysis, vol.4, September 1996, pp.89-105.