Discussion on Inexact Optimal Solution under Fuzzy Environment

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Abstract**—***The purpose of this paper is to explain that a convex combination of several partial solutions using a new criterion which did not solve the fuzzy problem. The main feature of this paper is twofold. First, we prepare a simple optimization problem to explain the previously proposed approach that considered criteria separately. Consequently, we conclude, previous results are partial solutions. Secondly, we study the same optimization problem as the one that appears in a previously published paper. After his new criterion is added, we solve the new fuzzy optimization problem to demonstrate that the previous solution is not the optimal one. Hence, the previously proposed approach is questionable and then his assertion of the meaninglessness of the exact optimal solution for the fuzzy problem cannot be treated as a valid statement. At last, we cite a paper that had referred to the questionable approach which had been improved by another published article to support our argument.*

Keywords—Fuzzy set, decision making, optimization, genetic algorithms, linear programming.

I. INTRODUCTION

Zadeh [1] introduced the concept of fuzzy set theory, and then Bellman and Zadeh [2] proposed the decision-making problem in a fuzzy environment. Zimmermann [3] applied the fuzzy set theory with proper membership functions to solve linear programming problems with several objective functions. Fuzzy set theory not only provided a mathematical way of representing imprecision or vagueness but also fuzzy linear programming has been applied to many practical areas such as production planning, resources allocation, transportation problems, and so on. Based on Zimmermann's method, the fuzzy objective functions, and fuzzy constraints and transferred to crisps ones by membership functions with the max-min operator. Hence, a unique exact optimal solution with the highest membership degree is derived. However, Wang [5] assumed that the exact optimal solution may not be desired by the decision-maker, owing to there being another criterion that is preferred by the decision-maker. He first found a family of inexact solutions, with an acceptable membership degree, that is obtained by a genetic algorithm with mutation along the weighted gradient direction. These solutions developed a convex set in the neighborhood of the exact optimal solution. Next, under the new criterion, he considered human-computer interaction to derive his *fuzzy optimal solution*. There are ten papers (e.g., Tang and Wang [6]; He et al. [8]; Ma et al. [9]; Tang et al.[10]; Van Hop [11]; Chiu et al. [13]; Lu et al. [14]; Baykasoglu and Gocken [15]) which have referred to Wang [5] in their references. However, none of them discovered the questionable results of Wang [5] that will be explained and revised in this paper. In this paper, we will carefully examine Wang's approach. After we add the new criterion preferred by the decision-maker into the fuzzy optimization problem, the result is that Wang's method will derive an unwarranted solution. We will demonstrate that his approach has a severe logic fault and that his assertion of "exact optimal solution in the fuzzy environment is meaningless" should be treated as invalid.

II. MATERIALS AND METHODS

The detailed description of Wang's approach, please refer to his paper. In the following, we just provide a very brief introduction. He mentioned that "However, the exact optimal solution may not be desired by the decision maker, because the membership function is not the preference criterion of the decision-maker. In general, the data are imprecise in a fuzzy environment, thus, it is meaningless to calculate an exact solution."

It means that there are two fuzzy optimization problems. We explicitly define as

a) The original fuzzy optimization problem, before the new criterion is added.

b) The new fuzzy optimization problem, after the new criterion, is added.

In the beginning, without the new criterion, Wang [5] considered the original fuzzy optimization problem to use a genetic method to obtain a family of acceptable solutions. Those solutions are in the neighborhood of the exact optimal solution for the original fuzzy optimization problem.

Next, a new criterion is added, so that Wang tried to solve the new fuzzy optimization problem. He studied the convex combination of two points in the family that satisfies the new criterion, and then he claimed that his solution is the fuzzy optimal solution.

III. RESULTS

3.1 Discussion of Wang's approach

Wang's solution is a convex combination of several solutions in the neighborhood of the original exact optimal solution (before the new criterion is added), different genetic computation will produce different results to constitute the family of inexact solutions with acceptable membership degree. Therefore, researchers following Wang's method will derive different fuzzy optimal solution depending on the results of the genetic algorithm. For fuzzy optimization problems, different researchers applied the same method to derive different results that seem quite suitable under imprecise condition. However, let us consider the following problem:

$$\max\ x-y \tag{1}$$

s.t. $x \geq 0$, $x \geq y$, and $x^2+y^2 \leq 1$.

The new criterion preferred by decision-maker is

$$y=0 \tag{2}$$

We will follow Wang's method to solve the above problem. If we assume that the acceptable solution has to satisfy $x-y \geq 0.5$. Next, we check the convex property of the constraint

$$x \geq y \tag{3}$$

That is, if (x_1, y_1) and (x_2, y_2) satisfy equation (3), then the convex combination $\lambda(x_1, y_1)+(1-\lambda)(x_2, y_2)$ with $0 \leq \lambda \leq 1$ also satisfies equation (3).

The convex property of the constraint

$$x^2+y^2 \leq 1, \tag{4}$$

can be verified as follows. We assume that (x_1, y_1) and (x_2, y_2) satisfy equation (4), then the convex combination is

$$\lambda(x_1, y_1)+(1-\lambda)(x_2, y_2)=(\lambda x_1+(1-\lambda)x_2, \lambda y_1+(1-\lambda)y_2) \tag{5}$$

We evaluate that

$$\begin{aligned}&(\lambda x_1+(1-\lambda)x_2)^2+(\lambda y_1+(1-\lambda)y_2)^2\\ &\qquad=\lambda^2(x_1^2+y_1^2)+(1-\lambda)^2(x_2^2+y_2^2)+2\lambda(1-\lambda)(x_1x_2+y_1y_2).\end{aligned} \tag{6}$$

We can consider $x_1x_2+y_1y_2$ as the inner product of (x_1, y_1) and (x_2, y_2), the owing to Schwarz's inequality, with $x_1^2+y_1^2 \leq 1$ and $x_2^2+y_2^2 \leq 1$, we know that

$$x_1x_2+y_1y_2 \leq \sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2} \leq 1 \tag{7}$$

so that we may rewrite equation (6) as

$$(\lambda x_1+(1-\lambda)x_2)^2+(\lambda y_1+(1-\lambda)y_2)^2 \leq \lambda^2+(1-\lambda)^2+2\lambda(1-\lambda)=1 \tag{8}$$

Hence, the convex combination also satisfies equation (4). Similarly, the constrain, x_1, x_2 $x \geq 0$, also has convex property.

Before the new criterion, equation (2) is added, for the original fuzzy optimization problem, applying some iterative process, among many results, for examples, we select $(x_1, y_1) = (0.995, 0.1)$ and $(x_2, y_2) = (0.954, -0.3)$ to create the family of inexact solutions.

Next, the new criterion, equation (2) is added, following Wang's approach, we need to solve the convex combination that satisfies the new criterion, that is

$$\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) = (x, 0) \tag{9}$$

to find that $\lambda = 0.75$ and

$$\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) = (0.985, 0) \tag{10}$$

Wang [5] believed that $(0.985, 0)$ is the fuzzy optimal solution by his approach.

We must point out that the new criterion of equation (2) should be obeyed by every researcher who tried to solve the original problem of equation (1). It is independent of the solution method. Hence, after the new criterion of equation (2) is given, then the optimization problem already changed to

$$\max\ x - y \tag{11}$$

s.t. $x \geq 0$, $x \geq y$, $y = 0$ and $x^2 + y^2 \leq 1$.

We may rewrite equation (10) as follows

$$\max\ x \tag{12}$$

s.t. $x \geq 0$ and $x^2 \leq 1$, to derive the exact optimal solution

$$x^* = 1 \text{ and } y = 0 \tag{13}$$

Wang [5] claimed that the result of equation (10) is the fuzzy optimal solution. However, the exact optimal solution of equation (13) is meaningless.

We have to say that Wang [5] forgot to realize that the new constraint must be used by every practitioner so the optimization problem already changed from equation (1) to equation (11), or simplified version, equation (12). Wang [5] insisted on using some partial solutions for the equation (1), and then considered the new criterion later to obtain his result. It means that in Wang's approach, criteria can be considered separately.

In the following, we will demonstrate in detail that this separate consideration of constraints will produce chaos. We will consider the previous problem in four steps.

First, we only consider

$$\max\ x - y \tag{14}$$

s.t. $x \geq 0$ and $x^2 + y^2 \leq 1$.

And then under some complicated computation to select two initial solutions, for examples, $(x_1, y_1) = (0.6, 0.8)$ and $(x_2, y_2) = (0.6, -0.8)$.

Second, we consider the new criterion, $y = 0$ to find $\lambda = 0.5$ and

$$\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) = (0.6, 0) \tag{15}$$

Third, we check whether or not the constraint, $x \geq y$ is satisfied, and then discover that $0.6 - 0 = 0.6 \geq 0$.

Fourth, we check the acceptable solution that has to satisfy $x-y\ge 0.5$. It shows that $0.6-0=0.6\ge 0.5$.

According to the above discussion, a fuzzy optimal solution, $(0.6,0)$, proposed by Wang [5] is obtained.

Before the convex combination, both $(x_1,y_1)=(0.6,0.8)$ and $(x_2,y_2)=(0.6,-0.8)$ are not satisfy the criterion of $x\ge y$ and $x-y\ge 0.5$. However, afterward, the convex combination, $(0.6,0)$ accidentally satisfies their two criteria. It indicates that by Wang's approach, some illogical solution can be the cornerstone of the next step derivation. It reveals that Wang's approach is not inexact but arbitrary.

The above example illustrates that Wang [5] committed a severe logic mistake in separately considering those constraints. Those constraints should be examined simultaneously.

3.2 A numerical example

In this section, we reconsider the numerical example of Wang [5], and we will point out the questionable aspects of the results of Wang [5].

A plant tries to produce two kinds of products, A and B, and the net profit of A and B are 6 and 4. The production of A and B needs labor time and a kind of main material. The labor and material requirements of each A are 2 labor unit time and 4 units, separately. Those of B are 3 labor unit time and 2 units. The available labor time is 70 units. By using overtime work, the plant can have an additional 30 units of labor time. The available material amount is 100 units, and there are 20 units safety storage of the material which is controlled by the general manager of this plant. The decision maker hopes that the objective profit reaches 200, at least not less than 160, and the overtime work and consumption of safety storage are not used too much. Hence, Wang [5] tries to consider the following fuzzy objective/resource optimization problem:

$$\max \quad \alpha \tag{16}$$

$$\text{s.t. } g_0(x_1,x_2)=6x_1+4x_2\ge 200-(1-\alpha)40 \tag{17}$$

$$g_1(x_1,x_2)=2x_1+3x_2\le 70+(1-\alpha)30 \tag{18}$$

$$g_2(x_1,x_2)=4x_1+2x_2\le 100+(1-\alpha)20 \tag{19}$$

$$x_1,x_2\ge 0,\ \alpha\in[0,1].$$

By the Zimmermann's tolerance approach, Wang mentioned that the exact optimal solution is $(x_1^*,x_2^*)=(20,15)$, $\mu_{\tilde{S}}(x_1^*,x_2^*)=0.5$ and the optimal value is 180. Wang [5] tried to use a genetic method with the mutation along the weighted gradient direction. The population size is $NG=100$, the acceptable membership degree is $\alpha=0.3$ to solve this problem. After 228 generations, Wang [5] claimed that the solutions $z2(0)$ and $z2(1)$ are better, and then he selected the labor constraint as the comparison criterion for $z2(0)$ and $z2(1)$. He mentioned that if the combination coefficient $\lambda=0.9116$ then there is no overtime. Wang [5] derived the combined solution

$$z=\lambda z2(1)+(1-\lambda)z2(0)=(25.1715,\ 6.5523) \tag{20}$$

the objective value, $g_0(z)=177.2383$, the membership function, $\mu_{\tilde{S}}(z)=0.31047$, and the constraints, $g_1(z)=70$, $g_2(z)=113.7907$.

Following Wang's [5] descriptions, the new condition is that there is no overtime. Therefore, practitioners should consider the following new fuzzy objective/resource optimization problem:

$$\max \quad \alpha \tag{21}$$

s.t. $g_0(x_1, x_2) = 6x_1 + 4x_2 \geq 200 - (1-\alpha)40$ (22)

$g_2(x_1, x_2) = 4x_1 + 2x_2 \leq 100 + (1-\alpha)20$ (23)

$x_1, x_2 \geq 0$, $\alpha \in [0,1]$.

IV. DISCUSSION

4.1 Our approach for the numerical example of Wang [5]

Our simple way to solve the above optimization is to assume that. $2x_1 + 3x_2 = A$, with $A \leq 70$. Hence, we replace x_2 by $(A - 2x_1)/3$ to rewrite the fuzzy objective/resource optimization problem:

$$\max \quad \alpha \tag{24}$$

s.t. $5x_1 + 2A \geq 240 + 60\alpha$ (25)

$4x_1 + A \leq 180 - 30\alpha$ (26)

$A \leq 70$, $0 \leq x_1 \leq 35$, $\alpha \in [0,1]$.

From the restriction of x_1, it yields that

$$\frac{240 + 60\alpha - 2A}{5} \leq x_1 \leq \frac{180 - 30\alpha - A}{4} \tag{27}$$

It follows that

$$20 + 130\alpha \leq A \leq 70 \tag{28}$$

and then

$$\alpha \leq \frac{5}{13} \approx 0.385 \tag{29}$$

We find the maximum possible of α under the new criterion.

It implies that if we take $\alpha = \frac{5}{13}$ and then solve the inequality of equation (28), then $A = 70$ so that it rewrites equations (27) as

$$\frac{320}{13} \leq x_1 \leq \frac{320}{13} \approx 24.615 \tag{30}$$

Hence g_0 and g_2 are both active, and then we obtain that $x_2 = 6.923$, $g_0(z) = 175.385$, $g_1 = 70$, $g_2(z) = 112.308$ and $\mu_{\tilde{S}}(z) = 0.385$.

Our solution attains the maximum value of α. On the other hand, Wang's [5] result, $\mu_{\tilde{S}}(z) = 0.31047$ did not attain the maximum value of α.

Our work reveals that Wang's approach did not solve the fuzzy objective/resource optimization problem under his new conditions. Consequently, based on Wang's inexact approach his claim of the meaninglessness of the exact optimal solution for fuzzy problems is questionable.

4.2 Our patchwork for another problem in Wang [5]

For completeness, we will prepare a patchwork to show another problem in Wang's inexact approach to finding the fuzzy optimal solution.

In our demonstration example, before the new criterion, equation (2), is added into the system, for the original fuzzy optimization problem (applying some iterative process, among many results), we select $(x_1, y_1) = (0.995, 0.1)$ and $(x_2, y_2) = (0.954, 0.3)$ to create the family of inexact solutions.

Next, the new criterion, equation (2) is added, following Wang's approach, we need to solve the convex combination that satisfies the new criterion, that is

$$\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) = (x, 0) \tag{31}$$

Under the condition $0 \le \lambda \le 1$, it yields that there is no solution for λ.

This illustrates an implicit problem in Wang's approach: Why are there points on the convex combination that satisfy the new criterion?

Next, we return to the example of Wang [5]. If the decision-maker hopes to achieve a big profit and more product B, then he will select the labor constraint as the comparison criterion for $z2(0)$ and $z2(2)$.

It means $z2(0) = (20.0293, 16.8886)$ and $z2(2) = (15.4291, 19.9664)$, with the convex combination

$$\lambda(z2(0)) + (1-\lambda)(z2(2)) = (15.4291 + 4.6001\lambda, 19.9664 - 3.0778\lambda) \tag{32}$$

with $0 \le \lambda \le 1$. However none of them satisfies the new criterion, that is

$$2(15.4291 + 4.6001\lambda) + 3(19.9664 - 3.0778\lambda) = 70 \tag{33}$$

Equation (33) has only one solution at $\lambda = 625.223$ that is not in the domain of $[0,1]$.

We may raise another question: if there are two solutions, λ_1 and λ_2, with $0 \le \lambda_1 < \lambda_2 \le 1$ so that both $\lambda_1(x_1, y_1) + (1-\lambda_1)(x_2, y_2)$ and $\lambda_2(x_1, y_1) + (1-\lambda_2)(x_2, y_2)$ satisfy the new criterion, then which one will be selected by Wang [5]?

The above example illustrates that in Wang's family of the preferred solutions for selection, if the decision-maker selects $z2(0)$ and $z2(2)$, then none of the convex combinations satisfy the new criterion. Wang did not tell us what the next step is when he failed to find $\lambda \in [0,1]$.

V. CONCLUSION

This paper has carefully examined Wang's [5] approach to solve a fuzzy optimization problem when a new criterion preferred by the decision-maker is added to the fuzzy optimization problem. The consequence is that Wang's method derives an unwarranted solution. Furthermore, it demonstrates that Wang's [5] approach has a severe logic fault, and did not solve the fuzzy problem under the new conditions. Moreover, we have shown how, when the new criterion is added to Wang's example, one can derive the correct optimal solution. At last, we recall that Van Hop [12] referred to Wang [5] to develop his solution process to solve linear programming problems under fuzziness and randomness environment. Chou et al. [16] pointed out that there are questionable results in the solution process of Van Hop [12] and then Chou et al. [16] provided a revision for Van Hop [12].

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