Discussion of two motivations provided by Sphicas 2006

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Abstract—*We discuss two motivations proposed by Sphicas (2006). First, after Lin (2019), we provide another partition for the feasible domain to show that there are at least three partitions to point out that the first motivation of Lin (2019) is not sufficient to support his solution procedure. For the second motivation of Sphicas (2006), we provide a detailed examination from the algebraic point of view to claim that his second motivation containing severe questionable results. We suggest researchers presenting a primitive algebraic method for inventory models with fixed and linear backorder costs.*

Keywords—Inventory models, algebraic method, fixed backorder cost, linear backorder cost.

I. INTRODUCTION

Sphicas[1] is the first paper to use the algebraic method to solve inventory models with fixed and linear backorder costs. Sphicas [1] applied a genuine method to partition the domain into two cases and then derived the optimal solution by algebraic methods. However, his genuine method is too sophisticated that is beyond imagined of ordinary practitioners, such that Cárdenas-Barrón[2], Chung and Cárdenas-Barrón[3], and Sphicas [4] provided further discussions for the same inventory models. Recently, Lin [5] mentioned that the algebraic approach provided by Sphicas [1] is too complicated for ordinary readers to absorb the motivation explained by Sphicas [1]. Consequently, Cárdenas-Barrón[2], Chung and Cárdenas-Barrón [3], and Sphicas [4] provided different algebraic to solve the same inventory model with linear and fixed backorder costs. Lin [5] pointed out Cárdenas-Barrón[2] containing several severe problems and then Lin [5] claimed that a primitive approach will be an interesting research topic for future researchers. In this paper, we will provide further discussion for the motivation proposed by Sphicas [1] for his algebraic method.

II. NOTATION AND ASSUMPTIONS

Our paper is focused on discussion with Sphicas [1]. To be compatible with Sphicas [1], we will adopt the same notation and assumptions that were used in his paper.

Notation

D is the demand rate per unit of time.

h is the holding cost per unit, per unit of time.

K is the ordering cost (setup cost) per order.

p is the backorder cost per unit, per unit of time (linear backorder cost).

Q is the order quantity.

$Q-S$ is the beginning inventory level, after backlogged quantity S.

r is an auxiliary expression, with $r=(h/p)$.

S is the backlogged amount.

TC is the total cost per unit of time.

π is the backorder cost per unit (fixed backorder cost).

Assumptions

1. The model is developed for only one product.
2. The inventory model is developed for an infinite planning horizon such that the goal is to minimize the first (and repeated) replenishment cycle.

3. The demand rate is constant over the entire planning horizon.
4. The shortages are allowed and fully backlogged.
5. There are two types of backorder costs: a linear backorder cost that is applied to average backorders per unit of time and a fixed cost that is applied to maximum backorder level without considering the backlogged waiting period.

III. A REVIEW FOR THE PARTITION OF THE DOMAIN OF SPHICAS [1]

In the following, we will provide a detailed discussion for the partition of the domain into $D\pi \geq \sqrt{2hDK}$ and $D\pi < \sqrt{2hDK}$.

If we observe the optimal solution derived by Sphicas [1] for the ordering quantity, EOQ_2^*, as

$$EOQ_2^* = \sqrt{\frac{2DK(h+p) - \pi^2 D^2}{hp}} \tag{3.1}$$

to imply a positive solution for EOQ_2^*, then we will imply the following condition:

$$2DK(h+p) > \pi^2 D^2 \tag{3.2}$$

In Sphicas [1], he divided the solution procedure into two cases: Case (A):

$$D^2\pi^2 \geq 2DKh \tag{3.3}$$

and Case (B):

$$2DKh > D^2\pi^2 \tag{3.4}$$

Here, we must point out that for Case (A), under the restriction $D^2\pi^2 \geq 2DKh$, Sphicas [1] mentioned that backlog is too expansive such that $S^* = 0$ and $Q^* = \sqrt{2DK/h}$ which is the traditional EOQ model without shortages.

For Case (B), under the restriction $2DKh > D^2\pi^2$, Sphicas [1] derived

$$S = \frac{hQ - \pi D}{h+p} \tag{3.5}$$

then he plugged the finding of (3.1) into (3.5) to ensure $S > 0$ that is $Q > (\pi D/h)$ to find that the condition of (3.2), to guarantee $Q > 0$, is not enough to imply the backorder quantity is positive. A stronger the condition

$$2DKh > \pi^2 D^2 \tag{3.6}$$

Appears that is the Case (B) proposed by Sphicas [1].

For ordinary readers to accept the $2DK(h+p) > \pi^2 D^2$ after (3.1) already derived that is reasonable, since from the numerator $2DK(h+p) - \pi^2 D^2$, to guarantee the numerator is positive to derive the condition of (3.2) will be understandable by researchers.

However, it is too difficult for ordinary readers to predict that $2DKh > D^2\pi^2$ of (3.4) in advance as proposed by Sphicas [1]. In Sphicas [1], he provided the first motivation for his approach as follows.

Based on Sphicas [1], the objective function for inventory models with two backorder costs: linear and fixed, is denoted as

$$TC(Q,S)=\frac{DK}{Q}+\frac{h(Q-S)^2}{2Q}+\frac{pS^2}{2Q}+\frac{DS\pi}{Q} \tag{3.7}$$

From (3.7), Sphicas [1] executed the following the derivation [1],

$$TC(Q,S)=\frac{h}{2Q}\left[(Q-S)^2+\frac{2DK}{h}\right]+\frac{pS^2}{2Q}+\frac{DS\pi}{Q} \tag{3.8}$$

To complete the square of $(Q-S)^2+\frac{2DK}{h}$, Sphicas [1] derived that

$$(Q-S)^2+\frac{2DK}{h}=\left[(Q-S)-\sqrt{\frac{2DK}{h}}\right]^2+2(Q-S)\sqrt{\frac{2DK}{h}} \tag{3.9}$$

such that (3.8) is rewritten as follows

$$TC(Q,S)=\frac{h}{2Q}\left[(Q-S)-\sqrt{\frac{2DK}{h}}\right]^2+\frac{pS^2}{2Q}+\frac{\left(D\pi-\sqrt{2DKh}\right)S}{Q}+\sqrt{2DKh} \tag{3.10}$$

Sphicas [1] mentioned that if $D^2\pi^2 \geq 2DKh$, then all terms in (3.10), with non-negative coefficient such that the optimal solution for S should be zero, as $S^*=0$, with $Q^*=\sqrt{2DK/h}$.

IV. A REVIEW FOR THE PARTITION OF THE DOMAIN OF LIN [5]

In Lin [5], he demonstrated that the rewriting of (3.7) is not unique. We recall the derivation of Lin (2009) since he rewrote (3.7) as follows

$$TC(Q,S)=\frac{1}{2Q}\left[pS^2+2DK\right]+\frac{h(Q-S)^2}{2Q}+\frac{DS\pi}{Q} \tag{4.1}$$

To complete the square of pS^2+2DK, Lin (2009) derived that

$$pS^2+2DK=\left[\sqrt{p}S-\sqrt{2DK}\right]^2-2\sqrt{2DKp}S \tag{4.2}$$

Such that (3.8) is rewritten as follows

$$TC(Q,S)=\frac{1}{2Q}\left[\sqrt{p}S+\sqrt{2DK}\right]^2+\frac{h(Q-S)^2}{2Q}+\frac{\left(D\pi-\sqrt{2DKp}\right)S}{Q} \tag{4.3}$$

If we observe (4.3), to make sure all coefficients are non-negative, then the next condition is derived,

$$D^2\pi^2 \geq 2DKp \tag{4.4}$$

V. OUR EXAMPLE TO PARTITION THE FEASIBLE DOMAIN

In this section, we will show another partition for the feasible domain to indicate there are at least three partitions for the feasible domain. We rewrite (3.7) as

$$TC(Q,S)=\frac{1}{2Q}\left[hQ^2+(h+p)S^2+2DK-2(hQ-\pi D)S\right] \tag{5.1}$$

under the restriction of $0 \leq S \leq Q$ and $0 < Q$.

If $hQ-\pi D\leq 0$, since $Q>0$, $S\geq 0$ and $-2(hQ-\pi D)\geq 0$, all terms in (5.1) are non-negative such that we derive $S=0$. The inventory model $TC(Q,S)$ is degenerated to the classical no shortage inventory model.

Based on the above discussion, we can divide the solution procedure into two cases: Case (i) $hQ\leq \pi D$, and Case (ii) $hQ>\pi D$. Hence, we construct another partition for the feasible domain. Owing to the partition for the feasible domain is not unique, the first motivation proposed by Sphicas [1] is not valid to convince researchers to accept his partition of (3.3) and (3.4).

VI. OUR DISCUSSION FOR THE SECOND MOTIVATION PROPOSED BY SPHICAS [1]

Lin [5] provided a comment on the second motivation for the solution approach of Sphicas [1]. However, Lin [5] considered the partial derivations in his discussion. Hence, we will not review the discussion mentioned by Lin [5] for the second motivation. In the following, we will provide our comments for the second motivation from an algebraic point of view.

We will present a further discussion for the second motivation proposed by Sphicas [1]. Sphicas [1] converted the objective function from

$$TC(Q,S)=\frac{DK}{Q}+\frac{h(Q-S)^2}{2Q}+\frac{pS^2}{2Q}+\frac{DS\pi}{Q} \tag{6.1}$$

to an indefinite expression,

$$TC=a_0+a_1(Q-A)^2+a_2(S-B)^2 \tag{6.2}$$

And Sphicas [1] claimed that "If it can be established that this can be done, if the coefficients are all nonnegative, and A and B are valid values for Q and S, then we can reach an immediate conclusion."

We recall that under the restriction $\pi D<\sqrt{2DKh}$, Sphicas [1] derived

$$TC=\frac{h}{h+p}(pK+\pi D)+\frac{hp}{2(h+p)Q}\left[Q-\sqrt{\frac{2DK(h+p)-\pi^2D^2}{hp}}\right]^2 + \frac{h+p}{2Q}\left[S-\frac{hQ-\pi D}{h+p}\right]^2, \tag{6.3}$$

such that we point out that

$$a_0=\frac{h}{h+p}(pK+\pi D) \tag{6.4}$$

$$a_1=\frac{hp}{2(h+p)Q} \tag{6.5}$$

$$a_2=\frac{h+p}{2Q} \tag{6.6}$$

$$A=\sqrt{\frac{2DK(h+p)-\pi^2D^2}{hp}} \tag{6.7}$$

and

$$B = \frac{hQ - \pi D}{h + p} \tag{6.8}$$

Based on our observation of (6.2-6.8), we can say that the motivation of Sphicas [1] is to transform TC from (3.1) to

$$TC = a_0 + \frac{a_1}{Q}(Q - A)^2 + \frac{a_2}{Q}(S - f(Q))^2 \tag{6.9}$$

where $f(Q)$ is an expression only in a variable Q, that will be $f(Q) = (hQ - \pi D)/(h + p)$.

Now, we assume that we know the final result of (6.9) in advance. We rewrite (6.9) as

$$TC = a_0 + \frac{a_1}{Q}\left(Q^2 - 2AQ + A^2\right) + \frac{a_2}{Q}\left(S^2 - 2f(Q)S + (f(Q))^2\right),$$

$$= a_0 + a_1 Q - 2a_1 A + a_1\frac{A^2}{Q} + a_2\frac{S^2}{Q} - 2a_2 f(Q)\frac{S}{Q} + \frac{a_2}{Q}(f(Q))^2. \tag{6.10}$$

On the other hand, we rewrite (6.1) as

$$TC = \frac{DK}{Q} + \frac{h}{2} - hS + \frac{p + h}{2Q}S^2 + \pi D\frac{S}{Q} \tag{6.11}$$

We compare (6.10) and (6.11) to imply that

$$a_2 = \frac{h + p}{2} \tag{6.12}$$

For those terms containing S, we obtain

$$-h + \frac{\pi D}{Q} = -2\left(\frac{h + p}{2}\right)\frac{f(Q)}{Q} \tag{6.13}$$

Based on (6.13), we derive the desired result of $f(Q)$ as

$$f(Q) = \frac{hQ - \pi D}{h + p} \tag{6.14}$$

We plug (6.12) and (6.14) into (6.1) to find

$$TC = \frac{h\pi D}{h + p} + \frac{2DK(h + p) - \pi^2 D^2}{2(h + p)Q} + \frac{hp}{2(h + p)}Q + \frac{h + p}{2Q}\left[S - \frac{hQ - \pi D}{h + p}\right]^2 \tag{6.15}$$

We rewrite the second and third terms of (6.15) to obtain

$$TC = \frac{h\pi D}{h + p} + \left(\sqrt{\frac{2DK(h + p) - \pi^2 D^2}{2(h + p)Q}}\right)^2 + \left(\sqrt{\frac{hp}{2(h + p)}Q}\right)^2 + \frac{h + p}{2Q}\left[S - \frac{hQ - \pi D}{h + p}\right]^2$$

$$= \frac{h\pi D}{h + p} + \left\{\left(\sqrt{\frac{2DK(h + p) - \pi^2 D^2}{2(h + p)Q}}\right) - \left(\sqrt{\frac{hp}{2(h + p)}Q}\right)\right\}^2$$

$$+2\left(\sqrt{\frac{2DK(h+p)-\pi^2D^2}{2(h+p)Q}}\right)\left(\sqrt{\frac{hp}{2(h+p)}}Q\right)+\frac{h+p}{2Q}\left[S-\frac{hQ-\pi D}{h+p}\right]^2$$

$$=\frac{h(pK+\pi D)}{h+p}+\left\{\left(\sqrt{\frac{2DK(h+p)-\pi^2D^2}{2(h+p)Q}}\right)-\left(\sqrt{\frac{hp}{2(h+p)}}Q\right)\right\}^2$$

$$+\frac{h+p}{2Q}\left[S-\frac{hQ-\pi D}{h+p}\right]^2,$$

$$=\frac{h}{h+p}(pK+\pi D)+\frac{hp}{2(h+p)Q}\left[Q-\sqrt{\frac{2DK(h+p)-\pi^2D^2}{hp}}\right]^2+\frac{h+p}{2Q}\left[S-\frac{hQ-\pi D}{h+p}\right]^2 \quad (6.16)$$

Our derivation of (6.16) is the result of (6.3) proposed by Sphicas [1].

If researchers know the final result of TC as (6.3) in advance, then following our above derivation, then the algebraic method proposed by Sphicas [1] becomes crystal clear.

The expression of (6.2) proposed by Sphicas [1] looks reasonable and intuitively acceptable. However, in fact, Sphicas [1] really needed the expression that should be expressed as (6.9).

However, for ordinary practitioners, unless you know the final result of (6.16) in advance, to accept the expression of (6.9) is questionable.

Therefore, the motivation of (6.2) provided by Sphicas [1] is not proper, the exact motivation should be revised to (6.9). Hence, we point out the second motivation provided by Sphicas [1] which is not sufficient to support the solution procedure in Sphicas [1].

VII. CONCLUSIONS

Even after Sphicas [1] provided two motivations for his algebraic approach, ordinary researchers still cannot understand the algebraic approach proposed by Sphicas [1]. Consequently, Cárdenas-Barrón[2], Chung and Cárdenas-Barrón [3], and Sphicas [4] provide further discussions for inventory models with fixed and linear backorder costs. Lin [5] discussed one motivation provided by Sphicas [1] and then we improve a typo in the discussion of Lin [5]. Moreover, we show that the second motivation proposed by Sphicas [1] containing more severe questionable results. Hence, we can claim that a primitive derivation for inventory models with fixed and linear backorder costs should be a hot research issue for academic society.

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