

Application of Convex and Concave Function

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Abstract— Convex function have an important role in many areas of mathematics. They are very crucial in the study of optimization problems where they are distinguished by a number of properties & application. For instance, a convex function on an open set has not more than one minimum point. Also in infinite dimensional spaces, for suitable additional hypothesis, convex function has to satisfy properties so as a result, they are almost well-understood functional in the calculus of variation. Concave functions play a important role in optimization. So-called algebraic concave functions are of similar importance in application. In this paper, we will discuss mathematical properties of concave & convex functions. Convex analysis has proved that in a certain sense, convexity is the next thing near to linearity. We will study, as soon as we allow inequalities in linear setting, it is immediately converging in the domain of convex analysis. Hence convexity arises naturally in the study of dual spaces hence there is a tendency to give the name "convexity" to anything that is compatible with variation minimization problems. In the other word, convexity and monotonicity are two sides of the same coin.

Keywords— Convex function, Concave upward, Concave downward, Inflexion points.

I. INTRODUCTION

The graph of a function f is said to be concave upward at a point $(p, f(p))$ if $f'(p)$ exists and if there is an open interval I containing p such that for all $x \in I$, $x \neq p$ the point $(x, f(x))$ on the graph is above the tangent line at $(p, f(p))$ to the curve where as the graph is said to be concave downward at a point $(p, f(p))$ if $f'(p)$ exists and if there is an open interval I containing p such that for all $x \in I$, $x \neq p$ the point $(x, f(x))$ lies below the tangent to the graph at $(p, f(p))$ to the curve.

Point of Inflexion: Those points where the curve changes from concave upwards to concave downwards or vice versa are called point of Inflexion.

Now concavity or convexity of a curve is determined from the sign of the second differential coefficient $\left(\frac{d^2y}{dx^2}\right)$. Therefore

If at P , $\left(\frac{d^2y}{dx^2}\right) > 0$ then it is concave upwards

If at P , $\left(\frac{d^2y}{dx^2}\right) < 0$ then it is concave downwards

At the point of inflexion, the curve changes from concave upwards to concave downwards or vice-versa, So $\left(\frac{d^2y}{dx^2}\right)$ changes sign and if the curve is continuous at the points, then $\left(\frac{d^2y}{dx^2}\right) = 0$

Hence the condition for point of inflexion are

(i) $\left(\frac{d^2y}{dx^2}\right) = 0$

(ii) the smallest positive integer n such that

$\left(\frac{d^{(n)}y}{dx^{(n)}}\right) \neq 0$ then n is odd at that point.

Convex and Concave functions

A plane or solid figure such as polygon or polyhedron is convex if the line segment joining any two point inside it lies wholly inside it. A function $f : I \rightarrow R$ is called convex if

$$f((1-\lambda)x + y) \leq (1-\lambda)f(x) + \lambda f(y);$$

for all x and y in I and all $\lambda \in [0; 1]$: If the inequality is reversed i.e

$$f((1-\lambda)x + y) \geq (1-\lambda)f(x) + \lambda f(y);$$

for all x and y in I and $\lambda \in [0; 1]$:

then f is said to be concave. If f is both convex and concave then f is said to be affine.

- 1) For what values of x is the curve $y=3x-2x^3$. Concave upwards and when is it concave downwards.

$$Y=3x^2-2x^2$$

$$\frac{dy}{dx} = 6x - 6x^2$$

$$\frac{d^2y}{dx^2} = 6 - 12x^3$$

$$\frac{d^2y}{dx^2} > 0 \rightarrow 6 > 12x$$

i.e $x < \frac{1}{2}$

∴ for $x < \frac{1}{2}$, the curve is concave upwards

Similarly,

$$\frac{d^2y}{dx^2} < 0 \rightarrow x > \frac{1}{2}$$

for $x > \frac{1}{2}$ the curve is concave downwards.

Find for what value of x is the curve $y=x^4 - 6x^3 + 12x^2 + 5x + 7$ concave upward/downward and find the point of inflection.

$$y=x^4 - 6x^3 + 12x^2 + 5x + 7$$

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 24x + 5$$

$$\frac{d^2y}{dx^2} = 12x^2 - 36x + 24$$

$$\frac{d^2y}{dx^2} > 0 \rightarrow 12x^2 - 36x + 24 > 0$$

$$\therefore x^2 - 3x + 2 > 0$$

i.e $(x-2)(x-1) > 0$

∴ $(x - 2 > 0 \text{ and } x - 1 > 0)$ or $(x - 2 < 0 \text{ and } x - 1 < 0)$

i.e.($x > 2$ and $x > 1$) or ($x < 2$ and $x < 1$)

$$\frac{d^2y}{dx^2} > 0 \Rightarrow x > 2 \text{ or } x < 1.$$

$$\frac{d^2y}{dx^2} < 0 \Rightarrow (x - 2)(x - 1) < 0.$$

$$\therefore (x - 2 < 0 \text{ and } x - 1 > 0) \text{ or } (x - 2 > 0 \text{ and } x - 1 < 0)$$

i.e.($x < 2$ and $x > 1$) or ($x > 2$ and $x < 1$)

$x > 2$ and $x < 1$ is not possible.

$$\frac{d^2y}{dx^2} < 0 \Rightarrow x < 2 \text{ and } x > 1 \text{ i.e. for } x \in (1, 2)$$

\therefore the curve is concave downward in (1,2)

\therefore the point of inflection are $x = 1$ and $x = 2$.

Proposition 1.1: Let f be a function which is differentiable some open interval containing p . Then (a) The graph of f is concave upward at $(p, f(p))$ if $f''(p) > 0$.

(b) The graph of f is concave downward at $(p, f(p))$ if $f''(p) < 0$.

Proof: (a) $f''(p) = \lim_{x \rightarrow p} \frac{f'(x) - f'(p)}{x - p}$ by definition. Since $f''(p) > 0$,

$$\lim_{x \rightarrow p} \frac{f'(x) - f'(p)}{x - p} > 0. \text{ Hence there is an open interval } I \text{ containing } p \text{ such that for all } x \in I, x \neq p, \text{ we have } \frac{f'(x) - f'(p)}{x - p} > 0 \dots\dots\dots (1)$$

Equation of tangent to the curve $(p, f(p))$ is

$$y = f(p) + (x - p) f'(p)$$

Let x be a number in the interval I such that $x \neq p$ and let Q be the point on the graph of f whose abscissa is x . Through Q draw a vertical line cutting the tangent in S . In order to prove that the graph of f concave upward at $(P, f(P))$ we must show that $f(x) - y_1 > 0$ where

$$y_1 = f(p) + (x - p) f'(p)$$

$$\text{now } f(x) - y_1 = f(x) - f(p) - (x - p) f'(p)$$

$$= f'(q)(x - p) - (x - p) f'(p) \text{ for some } q \text{ Between } x \text{ and } p$$

$$= (x - p)(f'(q) - f'(p)) \quad (2)$$

Since q is between x and p , it is in the interval I and so by taking $x = q$ in (1) we have

$$\frac{f'(q) - f'(p)}{q - p} > 0. \quad (3)$$

If $x > p$ then since q in between x and p we have $q > p$. From (3) then it follow that $f'(q) - f'(p) > 0$

and hence $f(x) - y_1 > 0$.

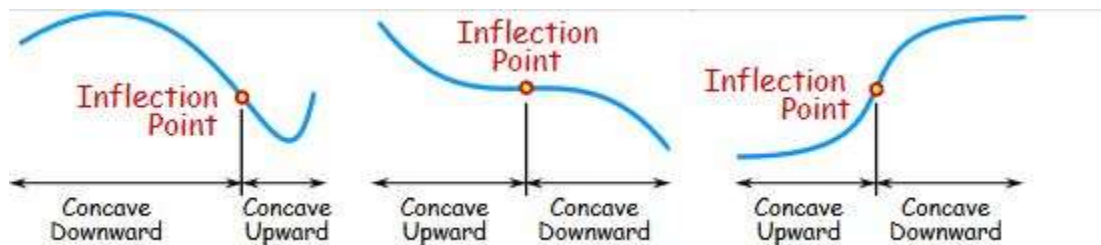
If $x < p$ we have $q < p$, in which case we infer from (3)

that $f''(q) - f''(p) < 0$ and hence $(x - p)(f''(q) - f''(p)) > 0$. It follows again that $f(x) - y_1 > 0$. Thus the graph of f is concave upward at $(p, f(p))$.

The converse of the above theorem is not true. For example if $f(x) = x^4$ the graph of f is concave upward at $(0,0)$ but $f''(0) = 0$.

If there is a point on the graph of a function at which the sense of concavity changes, then the graph crosses its tangent line at the point.

Such a point is called a point of inflection. The following figures show points of inflection.



Definition : A point, $(p, f(p))$ is a point of inflection of the graph of the function f if there is an open interval I containing p such that if x is in I then either.

- (i) $f''(x) < 0$ if $x < p$ and $f''(x) > 0$ if $x > p$. or
- (ii) $f''(x) > 0$ if $x < p$ and $f''(x) < 0$ if $x > p$.

The definition says nothing about $f''(p)$, but we have a theorem.

Proposition 1.2. If a function f is differentiable on some open interval containing p , then if $(p, f(p))$ is a point of inflection of the graph of f and if $f''(p)$ exists, $f''(p) = 0$.

Proof : Let $g(x) = f'(x)$, then $g'(x) = f''(x)$. Since $(p, f(p))$ is a point of inflection, $f''(x)$ changes sign at p and so $g'(x)$ changes sign at p . Therefore g has relative extremum at p . Since $g'(p) = f''(p)$ and since by hypothesis $f''(p)$ exists, it follows that $g'(p) = 0$ as p is a critical point of g . Therefore $f''(p) = 0$.

Example 1: Trace the curve of $y = x^3 - 3x^2 + x + 1$.

Solution.

$$f(x) = x^3 - 3x^2 + x + 1,$$

$$f'(x) = 3x^2 - 6x + 1,$$

$$f''(x) = 6x - 6,$$

$$f'''(x) = 6.$$

Intercept on Y -axis = 1. Intercepts on X -axis are 1, $1 + \sqrt{2}$ and $1 - \sqrt{2}$

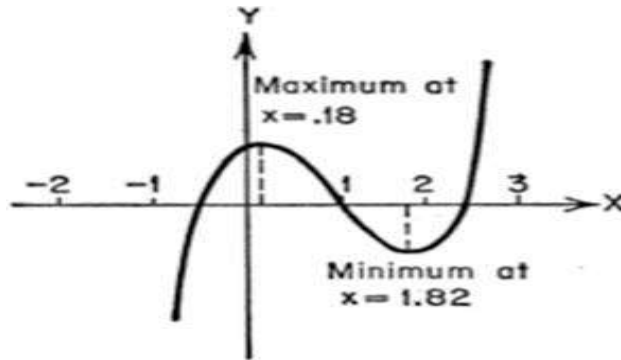
Tests show that the curve is not symmetric to the origin or to either axis.

Inspection shows that no values of x need be excluded.

Inspection shows no asymptotes.

Setting $f'(x)$ equal to zero and solving gives critical values at $x = \frac{3 \pm \sqrt{6}}{3}$. Testing these further, it is found that a minimum exists at $x = \frac{3 + \sqrt{6}}{3} = 1.82$; a maximum exists at $x = \frac{3 - \sqrt{6}}{3} = +0.18$.

Setting $f''(x) = 0$ and solving, $x = 1$; since $f'''(1) \neq 0$, there is a point of inflection at $x = 1$, i.e., the point (1,0). (That the abscissa of the point of inflection is also a root of $f(x) = 0$ is entirely accidental and of no significance.)



EXAMPLE 2. Trace the curve of $y = \frac{x}{x-1}$

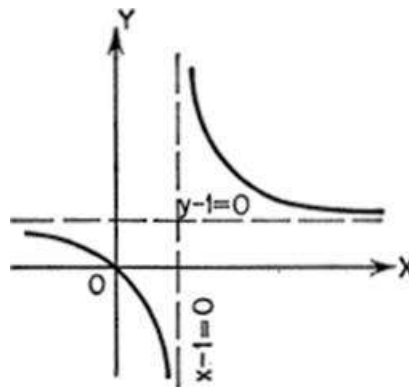
Intercept: $x = 0, y = 0$.

No symmetry.

Curve not defined when $x = 1$.

Asymptotes: $x - 1 = 0$, and $y - 1 = 0$.

$\frac{dy}{dx} = -\frac{1}{(x-1)^2}$, No values of x for which $f'(x) = 0$; hence no maxima or minima.



$f''(x) = 2(x-1)^{-3}$; no values of x for which $f''(x) = 0$; hence no points of inflection.

II. CONCLUSION

A concave function can more precisely capture the geometric boundary of a dataset than a convex function. Here, we propose a new algorithm to produce a concave function that is easy to understand and implement. The application area of the concave function is very wide. Convex functions are already widely used in geographic information processing, image processing, pattern recognition, and feature selection in machine learning areas. If the convex function is substituted with a concave function

in those tasks, increased performance or accuracy can be expected. Our contribution point is that we propose concave function algorithm for n-dimensional dataset whereas previous researches suggest for 2-dimensional data- sets.

Concaveness measure and graph is one of application of concave function. In the classification task, analysis of the dataset is important, but only basic statistical information can be obtained from a dataset if it is highly dimensional. If a concaveness graph is used, then information of geometric boundary can be obtained, and this information can be applied for planning classification tasks.

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