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An Applications of Fractional Calculus

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Abstract— *By considering different definations of fractional derivatives and fractional integrals, we study some examples, properties and also review some applications.*

Keywords— Fractional Derivative, Fractional Integral and Fractional Differential Equation.

I. INTRODUCTION

The concept of non-integral order of integration can be traced back to the genesis of differential calculus itself: the philosopher and creator of modern calculus G.W.Leibniz made some remarks on the the meaning and possibility of fractional derivative of order in the late 17th century. However a rigorous investigation was first carried out by Liouville in a series of papers from 1832-1837, where he defined the first outcast of an operator of fractional integration. Later investigations and further developments by among others Riemann led to the construction of the integral-based Riemann-Liouville fractional integral operator, which has been a valuable cornerstone in fractional calculus ever since. Prior to Liouville and Riemann, Euler took the first step in the study of fractional integration when he studied the simple case of fractional integrals of monomials of arbitrary real order in the heuristic fashion of the time; it has been said to have lead him to construct the Gamma function for fractional powers of the factorial . An early attempt by Liouville was later purified by the Swedish mathematician Holmgren , who in 1865 made important contributions to the growing study of fractional calculus. But it was Riemann who reconstructed it to fit Abel's integral equation, and thus made it vastly more useful. Today there exist many different forms of fractional integral operators, ranging from divided-difference types to infinite-sum types, but the Riemann-Liouville Operator is still the most frequently used when fractional integration is performed.

The fractional integral and differential is the most rapidly growing subject of mathematical analysis. The fractional integral and differential operator involving various special functions has found significant importance and applications in various sub fields of applicable mathematical analysis.

II. BASIC DEFINATIONS

(1) Riemann-Liouville Fractional Integral:

Let u be a non negative real number,f be piecewise continuous on , P'=(0,∞) and integrable on any finite subinterval of of P = [0,∞].Then for t>0, ${}_{c}D_{x}^{-u}$f(x) = $\frac{1}{\Gamma(u)}\int_{c}^{x}(x-t)^{u-1}$f(t)dt, u>0

is called Riemann-Liouville fractional integral of f of order u.

(2) Riemann-Liouville Fractional Derivative:

This definition can be defined using the definition of the fractional integral.Suppose u=n-v,where 0<u<1 and n is the smallest integer greater that v.Then the fractional derivative of f(x) of order v is D^{v}f(x) = $D^{n}[D^{-u}f(x)]$

(3) Laplace Transform Of the Fractional Integral:

If f(t) is a function of exponential order $\propto$, then $\int_{0}^{\infty}f(t)e^{-st}$dt. The Laplace transform is

L$\{f(t)\}$ = $\int_{0}^{\infty}f(t)e^{-st}$dt = F(s) .We defined

L$\{D^{-u}f(t)\}$ = $\frac{1}{\Gamma(u)}$L$\{t^{u-1}\}$L$\{f(t)\}$ = s^{-u}F(t), u>0

(4) Laplace Transform Of the Fractional Derivative:

The Laplace transform of $f^{(n)}$ is given by L$\{f^{(n)}\}$ = s^nF-$s^{n-1}f(0)$-$s^{n-2}f'$(0)……..f^{n-1}(0)

=s^nF(s)-$\sum_{k=0}^{n-1} s^{n-k-1} f^k(0)$

We have L$\{D^u f(t)\}$ = L$\{D^n[D^{-(n-u)}f(t)]\}$

= s^nL$\{[D^{-(n-u)}f(t)]\}$- $\sum_{k=0}^{n-1} s^{n-k-1} D^k[D^{-(n-u)}f(t)]$

= $s^n[s^{-(n-u)}F(t)]$- $\sum_{k=0}^{n-1} s^{n-k-1} D^{k-(n-u)}$f(0)

= s^uF(s)-$\sum_{k=0}^{n-1} s^{n-k-1} D^{k-n+u}$f(0)

In particular, if n=1 and n=2,

L$\{D^u f(t)\}$= s^uF(s)-$D^{-(1-u)}f(0)$, $0<u\leq 1$ and

L$\{D^u f(t)\}$= s^uF(s)-$sD^{-(2-u)}f(0)$-$D^{-(1-u)}f(0)$, $1<u\leq 2$

III. EXAMPLES OF FRACTIONAL INTEGRALS

Example 1 Lets evaluate $D^{-u}x^m$, where Re (u)>0,m>-1

$$D^{-u}x^m = \frac{1}{\Gamma(u)}\int_0^x (x-t)^{(u-1)}\, t^m \mathrm{dt}$$

$$= \frac{1}{\Gamma(u)}\int_0^x (1-\frac{t}{x})^{(u-1)} x^{u-1}\, t^m \mathrm{dt}$$

$$= \frac{1}{\Gamma(u)}\int_0^1 (1-v)^{(u-1)}\, x^{u-1}(xv)^m \mathrm{xdv}\text{ , (v=}\frac{t}{x}\text{)}$$

$$= \frac{1}{\Gamma(u)} x^{m+u} \int_0^1 v^m (1-v)^{(u-1)} \mathrm{dv}$$

$$= \frac{1}{\Gamma(u)} x^{m+u}\mathrm{B(m+1,u)}$$

$$= \frac{\Gamma(m+1)}{\Gamma(m+u+1)} x^{m+u}$$

Thus, $D^{-u}x^m = \frac{\Gamma(m+1)}{\Gamma(m+u+1)} x^{m+u}$, u>0, m>-1, x>0

Similarly, the fractional integral of a constant k of order u is $D^{-u}k = \frac{k}{\Gamma(u+1)} x^u$

, And in particular, if u=$\frac{1}{2}$, $D^{-\frac{1}{2}}x^0 = \frac{\Gamma(1)}{\Gamma(3/2)} x^{\frac{1}{2}} = 2\sqrt{\frac{x}{\pi}}$

$$D^{-\frac{1}{2}}x^1 = \frac{\Gamma(2)}{\Gamma(5/2)} x^{\frac{3}{2}} = \frac{4}{3}\sqrt{\frac{x^3}{\pi}}$$

$$D^{-\frac{1}{2}}x^2 = \frac{\Gamma(3)}{\Gamma(7/2)}x^{\frac{5}{2}} = \frac{16}{15}\sqrt{\frac{x^5}{\pi}}$$

Suppose f(t) = e^{at}, where a is constant, Then $D^{-u}e^{at} = \frac{1}{\Gamma(u)}\int_0^t (t-x)^{u-1}\, e^{at}$dx ,u>0

By the substitution y = t-x , then the above equation becomes

$$D^{-u}e^{at} = \frac{e^{at}}{\Gamma(u)}\int_0^t x^{u-1}\, e^{-ay}\, dx, \text{ u>0}$$

$D^{-u}e^{at} = E_t$(u,a) = $t^u E_{1,u+1}$(at)

Similarly, a direct application of the definition of the fractional integral,followed by some changes in veriables, results in

D^{-u}cos(at)= $\frac{1}{\Gamma(u)}\int_0^t x^{u-1}$cos[a(t-x)]dx = C_t(u,a), Re u>0

D^{-u}sin(at)= $\frac{1}{\Gamma(u)}\int_0^t x^{u-1}$sin [a(t-x)]dx = S_t(u,a), Re u>0

In particular, if u= $\frac{1}{2}$,

$D^{-1/2}e^{at} = E_t(\frac{1}{2}$,a) = $a^{-1/2}e^{at}Erf(at)^{1/2}$

$D^{-\frac{1}{2}}$cos(at)=$C_t(\frac{1}{2}$,a) = $\sqrt{\frac{2}{a}}$ [C(x)cos(at)-S(x)sin(at)]

$D^{-\frac{1}{2}}sin$(at) = $S_t(\frac{1}{2}$,a) = $\sqrt{\frac{2}{a}}$ [C(x)sin(at)-S(x)cos(at)]

Where x = $\sqrt{\frac{2at}{\pi}}$, C(x) = $\int_0^x cos(t^2)$dt, and S(x) = $\int_0^x sin(t^2)$dt,

In some cases,we may use simple trigonometric identities to calculate fractional integrals of some other trigonometric functions. For instance, using the identity

Cos(2t) = $2cos^2 t$ - 1 = 1-$2sin^2 t$

We have, $D^{-u}cos^2(at) = \frac{t^u}{2\Gamma(u+1)} + \frac{1}{2}\, C_t$(u,2a)

And $D^{-u}sin^2(at) = \frac{t^u}{2\Gamma(u+1)} - \frac{1}{2}\, C_t$(u,2a)

The Law of Exponents for Fractional Integrals

Theorem: Let f be continuous on J and let m,n>0.Then

$D^{-m}[D^{-n}f(t)]$ = $D^{-(m+n)}$f(t) = $D^{-n}[D^{-m}f(t)]$

Proof. By definition of fractional integral we have

$D^{-m}[D^{-n}f(t)]$=$\frac{1}{\Gamma(m)}\int_0^t (t-x)^{m-1}\,[D^{-n}f(t)]$dx

= $\frac{1}{\Gamma(m)}\int_0^t (t-x)^{m-1}\left[\frac{1}{\Gamma(n)}\int_0^x (x-y)^{n-1}\,f(y)\mathrm{dy}\right]$dx

= $\frac{1}{\Gamma(m)\Gamma(n)}\int_0^t (t-x)^{m-1}\,dx\int_0^x (x-y)^{n-1}\,f(y)\mathrm{dy}$

and $D^{-(m+n)}$f(t) = $\frac{1}{\Gamma(m+n)}\int_0^t (t-y)^{m+n-1}$ f(y)dy

= B(m,n) $\int_0^t (t-y)^{m+n-1}$ f(y)dy

Hence proved

IV. DERIVATIVES OF FRACTIONAL INTEGRAL AND THE FRACTIONAL INTEGRAL OF DERIVATIVES

For the fractional integral , $D^{-m}[D^{-n}f(t)] = D^{-n}[D^{-m}f(t)]$

We now develop a similar relation involving derivatives,However,generally

D$[D^{-n}f(t)] \neq D^{-n}[Df(t)]$

Theorem: Let f be a continuous on J and let n>0.If Df is continuous,then for all t>0

D$[D^{-n}f(t)] = D^{-n}[Df(t)]$+$\frac{f(0)}{\Gamma(n)}t^{n-1}$

Proof: By definition,

$D^{-n}f(t)$= $\frac{1}{\Gamma(n)}\int_0^t (t-z)^{n-1}\,f(z)$dz

If we make substitution z= t-x^{λ}, where $\lambda = \frac{1}{n}$, we obtain

$D^{-n}f(t)$= $\frac{1}{\Gamma(n)}\int_{t^n}^0 \left(x^{\lambda}\right)^{n-1}f\left(t-x^{\lambda}\right)\left(-\lambda x^{\lambda-1}\right)$dx

Which simplifies to

$D^{-n}f(t)$= $\frac{1}{\Gamma(n+1)}\int_0^{t^n} f(t-x^{\lambda})$ dx

Using the Leibniz's Integral Rule which states that

$\frac{d}{dt}\left[\int_0^{b(t)} f(t,x)dx\right]$ = f(t,b(t))b'(t)+$\int_0^{b(t)}\frac{\partial}{\partial t}$ f(t,x)dx

we then have

D$[D^{-n}f(t)]$ = $\frac{1}{\Gamma(n+1)}\left[f(0)nt^{n-1} + \int_0^{t^n}\frac{\partial}{\partial t}f(t-x^{\lambda})\,\mathrm{dx}\right]$

Now,we revise our substitution i.e let t-x^{λ}= z we obtain

$$\mathrm{D}[D^{-n}f(t)] = \frac{f(0)}{n\Gamma(n)}nt^{n-1} + \frac{1}{n\Gamma(n)}\int_t^0 \frac{\partial}{\partial t}f(z)(-\frac{1}{\lambda}x^{1-\lambda})\mathrm{dz}$$

Finally,since $\lambda=\frac{1}{n}$ and x = $(t-z)^{1/n}$ the preceding equation simplifies to

$$\mathrm{D}[D^{-n}f(t)] = \frac{f(0)}{\Gamma(n)}t^{n-1}+\frac{1}{\Gamma(n)}\int_0^t (t-z)^{n-1}\frac{\partial}{\partial t}f(z)\mathrm{dz}$$

Which implies that

$$\mathrm{D}[D^{-n}f(t)] = D^{-n}[Df(t)]+\frac{f(0)}{\Gamma(n)}t^{n-1}$$

V. EXAMPLES OF FRACTIONAL DERIVATIVES

Let evaluate the fractional derivative of f(t) = t^m of order n,where m≥ 0

By definition of fractional derivative

$$D^m \mathrm{f(t)} = D^r[D^{-n}f(t)]$$

Let m=r-n where 0<m<1. Then we have r=1 and m=1-n

$$D^m \mathrm{f(t)} = D^1\left[D^{-(1-n)}f(t)\right]$$

$$= D^1\left[D^{-(1-n)}t^m\right]$$

$$= D^1\left[\frac{\Gamma(m+1)}{\Gamma((m-n+1)+1)}t^{m-n+1}\right]$$

$$D^m t^m = (\mathrm{m-n+1})\frac{\Gamma(m+1)}{(m-n+1)\Gamma(m-n+1))}t^{m-n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}t^{m-n}$$

In particular, to find the ½ th order derivative of f(t) = t^m, for m = 0,1,2

$$D^{1/2}t^m = \frac{\Gamma(m+1)}{\Gamma(m-1/2+1)}t^{m-1/2}$$

$$D^{1/2}t^0 = \frac{\Gamma(1)}{\Gamma(1/2)}t^{-1/2} = \frac{1}{\sqrt{\pi t}},\ D^{1/2}t^1 = \frac{\Gamma(2)}{\Gamma(3/2)}t^{1/2} = 2\sqrt{\frac{t}{\pi}},\ D^{1/2}t^2 = \frac{\Gamma(3)}{\Gamma(5/2)}t^{3/2} = \frac{8}{3}\sqrt{\frac{t^3}{\pi}}$$

Now,suppose we wish to find m^{th} order fractional derivative of e^{at}, where 0<m<1

We have $D^m e^{at} = D^1[D^{-m}e^{at}]$

VI. EXAMPLES OF FRACTIONAL DIFFERENTIAL EQUATIONS

Example 1. Lets solve $D^{3/4}$y(t) = ay(t), where a is a constant.

Since 0<m=3/4≤ 1 so, by definition of laplace transform

$$\mathrm{L}\{D^{3/4}\mathrm{y(t)}\} = \mathrm{a\ L}\{y(t)\}$$

Which implies that $s^{3/4}$Y(s) - $D^{-(1-3/4)}$y(0) = a Y(s)

The constant $D^{-(1-3/4)}$y(0) = $D^{-1/4}$y(0) is the value of $D^{-1/4}$y(t) at t=0. If we assume this value exists, and call it c_1 then it becomes $s^{3/4}$Y(s) - c_1 = a Y(s)

Solving Y(s) we get, Y(s) = $\frac{c_1}{s^{3/4}-a}$

Y(t) = $L^{-1}\left\{\frac{c_1}{s^{3/4}-a}\right\} = c_1 t^{-1/4} E_{\frac{3}{4},\frac{3}{4}}\left(at^{3/4}\right)$.

VII. APPLICATIONS OF FRACTIONAL CALCULUS

The basic mathematical ideas of fractional calculus (integral and differential operations of non integer order)were developed long ago by the mathematicians Leibniz (1695), Liouville (1834), Riemann (1892), and others and brought to the attention of the engineering world by Oliver Heaviside in the 1890s, it was not until 1974 that the first book on the topic was published by Oldham and Spanier. Recent monographs and symposia proceedings have highlighted the application of fractional calculus in physics, continuum mechanics, signal processing, and electromagnetics. Here we state some of applications.

1. **Electric transmission lines**

During the last decades of the nineteenth century, Heaviside successfully developed his operational calculus without rigorous mathematical arguments. In 1892 he introduced the idea of fractional derivatives in his study of electric transmission lines. Based on the symbolic operator

form solution of heat equation due to Gregory(1846), Heaviside introduced the letter *p* for the differential operator and gave the solution of the diffusion equation

2. **Ultrasonic wave propagation in human cancellous bone**

Fractional calculus is used to describe the viscous interactions between fluid and solid structure. Reflection and transmission scattering operators are derived for a slab of cancellous bone in the elastic frame using Blot's theory. Experimental results are compared with theoretical predictions for slow and fast waves transmitted through human cancellous bone samples.

3. **Modeling of speech signals using fractional calculus**

In this paper, a novel approach for speech signal modeling using fractional calculus is presented. This approach is contrasted with the celebrated Linear Predictive Coding (LPC) approach which is based on integer order models. It is demonstrated via numerical simulations that by using a few integrals of fractional orders as basis functions, the speech signal can be modeled accurately.

4. **Modeling the Cardiac Tissue Electrode Interface Using Fractional Calculus**

The tissue electrode interface is common to all forms of biopotential recording (e.g., ECG, EMG, EEG) and functional electrical stimulation (e.g., pacemaker, cochlear implant, deep brain stimulation). Conventional lumped element circuit models of electrodes can be extended by

generalization of the order of differentiation through modification of the *Applications of fractional calculus* 1029 defining current-voltage relationships. Such fractional order models provide

an improved description of observed bioelectrode behaviour, but recent experimental studies of cardiac tissue suggest that additional mathematical tools may be needed to describe this complex system.

5. **Application of Fractional Calculus to the sound Waves Propagation in Rigid Porous Materials**

The observation that the asymptotic expressions of stiffness and damping in porous materials are proportional to fractional powers of frequency suggests the fact that time derivatives of fractional order might describe the behaviour of sound waves in this kind of materials, including relaxation

and frequency dependence.

6. **Using Fractional Calculus for Lateral and Longitudinal Control of Autonomous Vehicles**

Here it is presented the use of Fractional Order Controllers (FOC) applied to the path-tracking problem in an autonomous electric vehicle. A lateral dynamic model of a industrial vehicle has been taken into account to implement conventional and Fractional Order Controllers. Several control schemes with these controllers have been simulated and compared.

7. **Application of fractional calculus in the theory of viscoelasticity**

The advantage of the method of fractional derivatives in theory of visco elasticity is that it affords possibilities for obtaining constitutive equations for elastic complex modulus of viscoelastic materials with only few experimentally determined parameters. Also the fractional derivative method has been used in studies of the complex moduli and impedances for various models of viscoelastic substances.

8. **Fractional differentiation for edge detection**

In image processing, edge detection often makes use of integer-order differentiation operators, especially order 1 used by the gradient and order 2 by the Laplacian. This paper demonstrates how introducing an edge detector based on non-integer (fractional) differentiation can improve the criterion of thin detection, or detection selectivity in the case of parabolic luminance transitions, and the criterion of immunity to noise, which can be interpreted in term of robustness to noise in general.

VIII. CONCLUSION

In this paper , we have reviewed the definations of Riemann-Liouville Fractional Integral, fractional differential and fractional integral and differential Laplace operators. With the help of these definitions, we have achieved some important results. Also, we gave some applications and examples.

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