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The Study of Relation between Fourier Transform and Laplace Transform

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***Abstract*—** in this paper the author present the Laplace and Fourier transformations with their applications in solving boundary value problems of Heat conduction, Vibration of string, Transmission lines and also explain the relation between Laplace and Fourier transformation. The part of the course introduces two extremely powerful methods to solving differential equations that is the Fourier and the Laplace transforms. The choice of particular transform to be employed for the solution of an equation depends on the boundary conditions of the problem and the ease with which the transform can be inverted.

***Keywords*—** Fourier transform, Heat equation, Laplace transform, differential equation, boundary value problems

I. INTRODUCTION

Mathematics is everywhere in the world, it is used in every field. It is in every phenomenon, every technology, every observation, every experiment and more. All we need to do is to understand the logic hidden behind it and apply the Mathematics in it. The knowledge of Laplace transform and Fourier transform has in recent year becomes an essential part of mathematical background required of engineers and scientist. And the Laplace transform is used to find out the solutions of ordinary differtial equation whereas Fourier transform is used to find solution of ordinary differential equations**.**by using a particular type of definite integral as an operator a new function can be defined.one such operator is called Laplace transform. Laplace transform is used to solve the differential equation and boundary value problems. Fourier transforms transform a non-periodic function f (t) in time domain in to function f in frequency domain Fourier transform are highly useful in study of conduction of heat, wave propagation, communication

II. DEFINITIONS

The integral transform of a function f (t) denoted by I [f (t)], is defined by $\bar{f}(s)=\int_{t_1}^{t_2} f(t)\, k(s,t)\, dt$

Where k(s, t) is called the kernel of the transform and is a known function of s and t .the function f (t) is called the inverse transform of$\bar{f}(s)$.

Simple examples of a kernel are as follows:

(1) When k(s, t) = e^{-st}, it leads to the Laplace transform of f (t), i.e.

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt$$

(2) When k(s, t) =e^{ist}, we have the Fourier transform of f(x), i.e.

$$F(s) = \int_{-\infty}^{\infty} e^{ist} f(t)dt$$

III.RELATION BETWEEN FOURIER AND LAPLACE TRANSFORM

$$\text{If } f(t) = \begin{cases} e^{-xt} g(t), & t > 0 \\ 0 & , t < 0 \end{cases} \qquad \text{..............................} \ (1)$$

Then F {f (t)} =L {g (t)}

$$\text{We have } F\{f(t)\} = \int_{-\infty}^{\infty} e^{ist} f(t)dt \ = \int_{-\infty}^{0} e^{ist}.0\, dt + \int_0^\infty e^{ist}.e^{-xt} g(t)\, dt$$

$$= \int_0^\infty e^{(is-x)t}.g(t)\, dt$$

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$=\int_0^\infty e^{-pt}.g(t)\,dt$ where p=x-is

=L {g (t)}

Hence the Fourier transform of f (t) [defined by (1)] is the Laplace Transform of g(t).

IV.FOURIER TRANSFORM OF THE DERIVATIVE OF FUNCTION

The Fourier transform of the function u(x, t) is given by

$$F[u(x,t)] = \int_{-\infty}^{\infty} ue^{isx}.dx$$

Then the Fourier Transform of $\frac{\partial^2 u}{\partial x^2}$ i.e.

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{isx}\,dx = \left|e^{isx}\frac{\partial u}{\partial x} - is\,e^{isx}.u\right|_{-\infty}^{\infty} + (is)^2\int_{-\infty}^{\infty} ue^{isx}\,dx$$

On applying the general rule of integration by parts. If u and $\frac{\partial u}{\partial x}$ tend to zero as x tends to $\pm\infty$,

Then $F\left[\frac{\partial^2 u}{\partial x^2}\right] = -s^2 F[u]$ ………… (1)

Similarly in the case of Fourier sine and cosine transforms, we have

$F_s\left[\frac{\partial^2 u}{\partial x^2}\right] = s(u)_x = 0 - s^2 F_s[u]$ And ………….. (2)

$F_c\left[\frac{\partial^2 u}{\partial x^2}\right] = -\left(\frac{\partial u}{\partial x}\right)_{x=0} - s^2 F_c[u]$ ..…………. (3)

In general, the Fourier Transform of the nth derivative of f(s) is given by

$F\left[\frac{\partial^n u}{\partial x^n}\right] = (-is)^2 F[f(x)]$ …………… (4)

Provided the first n-1 derivatives vanish as $x \rightarrow \pm\infty$

For $F[f^n(x)] = \int_{-\infty}^{\infty} f^n(x)e^{isx}.dx$

By the general rule of integration by parts, when follows (4)

V.INVERSE LAPLACE TRANSFORM BY METHOD OF RESIDUES

Let the Laplace transform of $f(x)$ be $\bar{f}(s)$ so that

$$\bar{f}(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Under certain conditions therefore

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$$f(x) = \lim_{r\to\infty} \frac{1}{2\pi i}\int_c e^{xs}\bar{f}(s)dx$$

=Sum of the Residues of $e^{xs}\bar{f}(s)$ at the poles of $f(s)$

(1)Evaluate $L^{-1}\left\{\frac{1}{(s-1)(s^2+1)}\right\}$ by the method of residue

Sol:-

$$\text{Since } \left|\frac{1}{(s-1)(s^2+1)}\right| - \left|\frac{1}{s^3}\right| \text{ for} |s| \to \infty$$

$$\textbf{Therefore,} L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = sum\ of\ Res\left[\frac{e^{xs}}{(s-1)(s^2+1)}\right] \textbf{ at the poles } s = 1, \pm i$$

$$\textbf{Now } (Res)_s = 1 = \lim_{s\to 1}\left[\frac{(s-1)e^{xs}}{(s-1)(s^2+1)}\right] = \frac{e^x}{2}$$

$$(Res)_{s=i} = 1 = \lim_{s\to i}\left[\frac{(s-i)e^{xs}}{(s-1)(s^2+1)}\right] = \frac{e^{ix}}{-2(i+1)}$$

Changing I to –I we get

$$(Res)_{s=-i} = 1 = \lim_{s\to -i}\left[\frac{(s+i)e^{xs}}{(s-1)(s^2+1)}\right] = \frac{e^{ix}}{-2(1-i)}$$

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \frac{e^x}{2} - \frac{1}{2}\left(\frac{e^{ix}}{(i+1)} + \frac{e^{-ix}}{(1-i)}\right)$$

$$= \frac{1}{2}(e^x - sinx - cosx)$$

VI.APPLICATION OF TRANSFORM TO BOUNDARY VALUE PROBLEMS

In one dimensional boundary value problems, the partial differtial equation can easily be transformed into an ordinary differtial equation by applying a suitable transform .the required solution is then obtained by solving this equation and inverting by means of complex inversion formula or by any other method .In two dimensional problems, it is sometimes required to apply the transforms twice and the desired solution is obtained by double invasion.

1) If in a problem $u(x,t)_{x=0}$ is given then we use infinite sine transform to remove $\frac{\partial u^2}{\partial x^2}$ from the differtial equation
   In case$[\partial u(x,t)/\partial x]_{x=0}$ is given then we employ infinite cosine transform to remove $\frac{\partial u^2}{\partial x^2}$
2) If in a problem u(0,t) and u(l ,t) are given ,then we use finite sine transform to remove $\frac{\partial u^2}{\partial x^2}$ from the differtial equation
   In case$[\partial u/\partial x]_{x=0}$ and $[\partial u/\partial x]_{x=l}$ is given then we employ infinite cosine transform to remove $\frac{\partial u^2}{\partial x^2}$
   The method of solution is best explained through the following Examples

VII.HEAT CONDUCTION, VIBRATION OF STRING, TRANSMISSION LINES

1) **Determine the distribution of temperature in the semi-infinite medium $x \geq 0$,when the end x=0**

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Is maintained at zero temp. And the initial distribution of temperature is f(x)

Sol:-

Let u(x,t) be the temp. At any point x and at any time t .We have to solve the Heat flow equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (x > 0, t > 0) \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$subject\ to\ the\ initial\ condition \quad u(x,0) = f(x) \qquad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

And the boundary condition $u(0,t) = 0$ ………………………….. (3)

Taking Fourier sine transform and denoting $F_s[u(x,t)]\ by\ \bar{u}_s$ we have

$$\frac{d\overline{u_s}}{dt} = c^2[su(0,t) - s^2\overline{u_s}] \quad (x > 0, t > 0) \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Also Fourier sine transform of (2) is $\overline{u_s} = \overline{f_s}(s)$ at t=0

Solving (4) and (5) we get $\overline{u_s} = \overline{f_s}(s)e^{-c^2s^2t}$

Hence taking its inverse Fourier sine transform we obtain

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \overline{f_s}(s)e^{-c^2s^2t}\, sinxs\, ds$$

2) **An infinite string is initially at rest and that initial displacement is y(x,0) $(-\infty < x < \infty)$ determine the displacement y(x, t) of string**

Sol:-

The equation for vibration of string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \ (1)$$

And the initial conditions are $\left(\partial y/_{\partial t}\right)_{t=0} = 0\ ; y(x,0) = f(x)$ ……………………….. (2)

Multiplying (1) by e^{isx} and integrating w.r.t x from $-\infty\ to\ \infty$

$$\frac{\partial^2 Y}{\partial t^2} = c^2(-s^2 Y) \quad provided\ that\ \ {\partial y}/_{\partial x} \to 0\,, x \to \infty$$

The solution is $\frac{\partial^2 Y}{\partial t^2} + c^2(s^2 Y) = 0\ \ is\ \ Y = A_1 cos\ cst + A_2 sin\ cst$ …………………………….. (3)

Also Fourier transform of (2) are ${\partial y}/_{\partial t} = 0\ and\ Y = F(s) \quad when\ t = 0$

Applying these to (3) we get $A_2 = 0 \quad and \quad A_1 = F(s)$

$$Y = F(s)\, cos\ cst$$

Now taking the Fourier transform we get

$$y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s) coscst\, e^{isx}\, dx$$

$$y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\frac{e^{icst} + e^{-icst}}{2} e^{isx}\, dx$$

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$$y(x,t) = \frac{1}{4\pi}\int_{-\infty}^{\infty} F(s)\left(e^{icst} + e^{-icst}\right)e^{isx}\,dx$$

$$y(x,t) = \frac{1}{4\pi}\int_{-\infty}^{\infty} F(s)e^{-is(x-ct)} + F(s)e^{-is(x+ct)}\,dx$$

$$y(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] \quad \text{Where} \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx)}\,dx$$

3) A semi- infinite transmission line of negligible inductance and leakance per unit length has it's voltage and current equal to zero a constant voltage V_0 is applied at sending end (x=0) at t=0.find the voltage and current at any point x and at any instant

Sol:-

Let v(x,t) and i(x,t) be the voltage and current at any point x and at any time t

If L=0 and G=0 then

The transmission line equation becomes $\frac{\partial v}{\partial x} = -Ri, \frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t}$

i.e. $\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t}$ ……………………… (1)

The boundary condition are v (0, t) = V_0 and i(x,t) is finite for all x and t.

The initial condition are v(x, 0) =0, i(x, 0) =0 ………………………. (2)

Laplace transform of (1) are

$$\frac{d^2\bar{v}}{dx^2} = RC(s\bar{v} - 0)$$

$$\frac{d^2\bar{v}}{dx^2} - RC(s\bar{v}) = 0 \qquad \text{……………………. (3)}$$

The Laplace transform of condition in (2) are

$$\bar{v}(0,s) = \frac{v_0}{s} \quad at\ x = 0 \qquad \text{… ….……………………. (4)}$$

And $\bar{v}(x,s)$ remains finite as $x \to \infty$

The solution of (3) is

$$\bar{v}(x,s) = C_1 e^{\sqrt{RCs}x} + C_2 e^{-\sqrt{RCs}x} \qquad \text{………………………. (5)}$$

To satisfy condition (5) we must have $C_1 = 0$

Using the condition (4) we get $C_2 = \frac{v_0}{s}$ thus

$$\bar{v}(x,s) = \frac{v_0}{s}e^{-\sqrt{RCs}x}$$

Using the inversion formula we obtain

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$$v(x,t)= v_0 L^{-1}\left\{\frac{e^{-\sqrt{RC}x\sqrt{s}}}{s}\right\} = v_0 erfc\left(x\frac{\sqrt{Rc}}{2\sqrt{t}}\right)$$

$$= v_0\left(x\frac{\sqrt{Rc}}{2\sqrt{x}}\right)\int_0^t u^{-3/2} e^{-(Rcx^2/4u)}\,du$$

Since $i = -\frac{1}{R}\frac{\partial v}{\partial x}$ we obtained by differtiation

$$i(x,t) = v_0\left(x\frac{\sqrt{\frac{C}{R}}}{2\sqrt{x}}\right)t^{-3/2} e^{-(Rcx^2/4t)}$$

VII.CONCLUSION

Through this paper we give the relation between Laplace transform and Fourier Transform and explain it by using examples of both the transforms. Laplace transform have wide use in many field so in this paper we use Laplace Transform to solve residues and Fourier Transform for the derivative of function. And explain the boundary value problem of Transforms for Heat conduction, Vibration of string, Transmission lines.

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